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An Enhanced Euler Characteristic of Sutured Instanton Homology

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For a balanced sutured manifold (M,γ), we construct a decomposition of $SHI(M,\gamma)$ with respect to torsions in $H = H_1(M;\mathbb{Z})$, which generalizes the decomposition of $I^\sharp(Y)$ in a previous work of the authors. This decomposition can be regarded as a candidate for the counterpart of the torsion spin^c decompositions in $SFH(M,\gamma)$. Based on this decomposition, we define an enhanced Euler characteristic $\chi_{\rm en}(SHI(M,\gamma)) \in \mathbb{Z}[H]/\pm H$ and prove that $\chi_{\rm en}(SHI(M,\gamma)) = \chi(SFH(M,\gamma))$. This provides a better lower bound on $\dim_{\mathbb{C}} SHI(M,\gamma)$ than the graded Euler characteristic $\chi_{\rm gr}(SHI(M,\gamma))$. As applications, we prove instanton knot homology detects the unknot in any instanton L-space and show that the conjecture $KHI(Y,K) \cong \widehat{HFK}(Y,K)$ holds for all $(1,1)$-L-space knots and constrained knots in lens spaces, which include all torus knots and many hyperbolic knots in lens spaces.

1 Introduction

Sutured instanton Floer homology was introduced by Kronheimer and Mrowka in [36]. It combines instanton Floer homology with sutured manifold theory and has become a powerful tool since then. In [41], Ghosh and the first author constructed a decomposition of the sutured instanton Floer homology $SHI(M,\gamma)$ of a balanced sutured manifold with respect to the group $(H_2(M,\partial M;\mathbb{Z}))^* \cong H_1(M;\mathbb{Z})/\mathrm{Tors}$. More precisely, a

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basis of $H_2(M, \partial M; \mathbb{Z})$ induces a multi-grading on $SHI(M, \gamma)$, which can be identified with the group $H_1(M; \mathbb{Z})/\mathrm{Tors}$. In [42], the authors of the current paper studied the Euler characteristics of this decomposition of $SHI(M, \gamma)$ and related it to the Euler characteristic of $SFH(M, \gamma)$, which is known as the sutured Floer homology introduced by Juhász, and whose Euler characteristic has been understood by work of Friedle, Juhász, and Rasmussen in [18]. The study of Euler characteristics was further used by the author to compute the instanton Floer homology of some families of $(1, 1)$-knots in a general lens space and was recently further utilized by Xie and Zhang [64] to prove that links in S^3 all admit irreducible $SU(2)$ representations except for connected sums of Hopf links.

However, only having the decomposition of $SHI(M, \gamma)$ along the group $H_1(M; \mathbb{Z})/\mathrm{Tors}$ is not fully satisfactory for the following two reasons.

(1) Among all known Floer homology theories for sutured manifolds, we have known that sutured monopole Floer homology is isomorphic to sutured Floer homology by the work of Lekili [39] and Baldwin and Sivek [11], and it is conjectured that the sutured instanton Floer homology is also isomorphic to sutured Floer homology by Kronheimer and Mrowka [36]. However, sutured Floer homology decomposes along spin^c structures and, in particular, the first Chern classes of torsion spin^c structures have Poincaré dual in the torsion part of the group $H_1(M; \mathbb{Z})$, so there should be some corresponding decomposition of sutured instanton Floer homology.

(2) The original decomposition along $H_1(M; \mathbb{Z})/\mathrm{Tors}$ collapses all torsion parts into a single summand of $SHI(M, \gamma)$, and some information may lost in this collision; see Example 1.4.

In this paper, in order to solve this problem coming from collapsing torsion parts, we obtain the following.

Theorem 1.1 (Main theorem). Suppose (M, γ) is a balanced sutured manifold and $H = H_1(M; \mathbb{Z})$. Then there is a (possibly noncanonical) decomposition

$$SHI(M, \gamma) = \bigoplus_{h \in H} SHI(M, \gamma, h).$$

This decomposition depends on some auxiliary choices. In particular, it is defined up to a global shift of H. We define the **enhanced Euler characteristic** of SHI by

$$\chi_{\mathrm{en}}(SHI(M, \gamma)) := \sum_{h \in H} \chi(SHI(M, \gamma, h)) \cdot h \in \mathbb{Z}[H]/ \pm H.$$

Then we have

$$\chi_{\mathrm{en}}(SHI(M,\gamma)) = \chi(SFH(M,\gamma)) \in \mathbb{Z}[H]/\pm H. \tag{1.1}$$

The similar results also hold for $SHM(M,\gamma)$.

Remark 1.2. If $H_1(M;\mathbb{Z})$ has no torsion, then the decomposition in Theorem 1.1 is just induced by the multi-grading mentioned above and Equation (1.1) reduces to [42, Theorem 1.2]. By results in [18], we have $\chi(SFH(M,\gamma)) = \tau(M,\gamma)$, where $\tau(M,\gamma)$ is a (Turaev-type) torsion element that can be calculated by Fox calculus. In particular, if ∂M consists of tori and γ consists of two parallel copies of a curve m_i with opposite orientations on each boundary component, by [18, Proof of Lemma 6.1] and [57, Proposition 2.1], we have

$$\tau(M,\gamma) = \tau(M)\cdot\prod_i([m_i]-1),$$

where $\tau(M)$ is the Turaev torsion of M [60].

Though the decomposition in Theorem 1.1 has not been proved to be canonical, we expect it to be well-defined up to a global grading shift of H. The following theorem indicates this decomposition is a candidate for the counterpart of the spin^c decomposition. The proof is essentially due to [11, 39].

Theorem 1.3. Suppose (M,γ) is a balanced sutured manifold. Then there is a spin^c structure $\mathfrak{s}_0 \in \mathrm{Spin}^c(M,\gamma)$ such that for any $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$, we have an isomorphism

$$SHM(M,\gamma,PD(\mathfrak{s}-\mathfrak{s}_0)) \cong SFH(M,\gamma,\mathfrak{s})\otimes\Lambda,$$

where $\mathrm{PD} : H^2(M,\partial M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is the Poincaré duality map and Λ is the Novikov ring.

For an element in a group ring $\mathbb{Z}[G]$

$$x = \sum_{g\in G} c_g\cdot g,\ \text{for } c_g\in\mathbb{Z},$$

define

$$\|x\| = \sum_{g\in G}|c_g|.$$

This definition is still well-defined for an element in $\mathbb{Z}[G]/\pm G$. By construction of Euler characteristics, we have

$$\dim_{\mathbb{C}} SHI(M,\gamma) \geq \|\chi_{\mathrm{en}}(SHI(M,\gamma))\| \geq \|\chi_{\mathrm{gr}}(SHI(M,\gamma))\|. \tag{1.2}$$

To provide an example that the second inequality in (1.2) is not always sharp, and hence χ_{en} contains more information than χ_{gr}, we consider an example from constrained knots studied by the second author [66].

Example 1.4. Consider the 1-cusped hyperbolic manifold $M = m006$ in the *Snappy* program [16]. We have $H_1(M;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_5 \cong \mathbb{Z}\langle t,r\rangle/(5r)$. By the list of constrained knots in [65], Dehn filling along the slope $(1,0)$ (in the basis from *Snappy*) gives the lens space $L(5,3)$ and the core knot is the constrained knot $C(5,3,4,3,1)$. Suppose γ consists of two parallel copies of the curve of slope $(1,0)$. Then we have

$$\tau(M,\gamma) = 1 + r + t + rt + r^2t - r^3t - r^4t + rt^2 + r^2t^2,$$

and

$$\tau(M,\gamma)|_{r=1} = 1 + 1 + t + t + t - t - t + t^2 + t^2 = 2 + t + 2t^2.$$

Hence, we have

$$\|\chi_{\mathrm{en}}(SHI(M,\gamma))\| = \|\tau(M,\gamma)\| = 9 \text{ and } \|\chi_{\mathrm{gr}}(SHI(M,\gamma))\| = \|\tau(M,\gamma)|_{r=1}\| = 5.$$

Suppose K is a knot in a closed 3-manifold Y. Let

$$Y(1) := Y\backslash B^3 \text{ and } Y(K) := Y\backslash \mathrm{int} N(K).$$

Suppose δ is a simple closed curve on $\partial Y(K) \cong S^2$, and suppose γ_K is two copies of the meridian of K with opposite orientations. In Heegaard Floer theory, we have

$$SFH(Y(1),\delta) \cong \widehat{HF}(Y) \text{ and } SFH(M,\gamma_K) \cong \widehat{HFK}(Y,K),$$

where $\widehat{HF}(Y)$ and $\widehat{HFK}$ are the hat versions of the Heegaard Floer homology and the knot Floer homology defined by Ozsváth and Szabó [48, 50, 56]. In instanton theory, Kronheimer and Mrowka [36] defined

$$I^\sharp(Y) := SHI(Y(1),\delta) \text{ and } KHI(Y,K) := SHI(Y(K),\gamma_K).$$

An application of Theorem 1.1 is the following unknot detection result in rational homology spheres.

Theorem 1.5. Suppose K is a null-homologous knot in a rational homology sphere Y. If

$$\dim_{\mathbb{C}} I^{\sharp}(Y) = |H_1(Y;\mathbb{Z})|, \tag{1.3}$$

then K is the unknot that is, it bounds a disk in Y, if and only if

$$\dim_{\mathbb{C}} KHI(Y,K) = \dim_{\mathbb{C}} I^{\sharp}(Y). \tag{1.4}$$

Remark 1.6. Since instanton theory is closely related to $SU(2)$ representations of fundamental groups, Theorem 1.5 may be used to show that for any nontrivial null-homologous knot K in a rational homology sphere Y, the fundamental group $\pi_1(Y(K))$ admits an irreducible representation in $SU(2)$ such that the meridian of K is mapped to a traceless element in $SU(2)$. However, the authors do not know how to prove the nondegeneracy results similar to [10, Section 4.1] for generators of $KHI(Y,K)$.

Remark 1.7. Rational homology spheres that satisfy (1.3) are called **instanton L-spaces**. Theorem 1.5 cannot be generalized to knots that are not null-homologous because simple knots in lens spaces also satisfy (1.4) [43, Proposition 1.9]. It is a natural conjecture that simple knots are the only knots in lens spaces satisfy (1.4) (For Heegaard Floer theory, see [2, Conjecture 1.5]).

Following the similar strategy, we can prove the following theorem for knots whose *KHI* have small dimensions.

Theorem 1.8. Suppose K is a null-homologous knot in a rational homology sphere Y. If

$$\dim_{\mathbb{C}} KHI(Y,K) = \dim_{\mathbb{C}} I^{\sharp}(Y) + 2 = |H_1(Y;\mathbb{Z})| + 2, \tag{1.5}$$

then K must be a genus-one-fibred knot.

Remark 1.9. The only knots in S^3 satisfying (1.5) are the trefoil and its mirror. Hence, Theorem 1.8 is a generalization of the trefoil detection result in S^3 [12, Theorem 1.6]. Both proofs are based on the nonvanishing result on the “next-to-top” grading [12, Theorem 1.7] for fibred knots.

Remark 1.10. For a knot K in an instanton L-space Y with

$$\dim_{\mathbb{C}} KHI(Y,K) = \dim_{\mathbb{C}} I^{\sharp}(Y) + 4 = |H_1(Y;\mathbb{Z})| + 4,$$

we may still conclude K is fibred using the same strategy. However, it is impossible to pin down the genus because there are at least two knots in S^3 with different genera: the figure-8 knot with genus one and the $T_{(2,5)}$ torus knot with genus two. Recently, there are many new results [5, 45, 46] about the Khovanov homology and the knot Floer homology of $T_{(2,5)}$. It is an interesting question that if these results can be applied to instanton knot homology.

Remark 1.11. We can prove similar detection results in Heegaarders Floer theory; see Section 6.

Another application of Theorem 1.1 is to compute $KHI(Y,K)$ for all $(1,1)$-L-space knots and constrained knots in lens spaces. The calculation is based on the following theorem.

Theorem 1.12 ([43, Theorem 1.6], see also [6, Theorem 1.1]). Suppose $K \subset Y$ is a $(1,1)$-knot in a lens space (including S^3). Then we have

$$\dim_{\mathbb{C}} KHI(Y,K) \leq \dim_{\mathbb{F}_2} \widehat{HFK}(Y,K).$$

Corollary 1.13. Suppose $K \subset Y$ is a $(1,1)$-knot in a lens space (including S^3). If K is either a L-space knot, or a constrained knot, then

$$\dim_{\mathbb{C}} KHI(Y,K) = \dim_{\mathbb{F}_2} \widehat{HFK}(Y,K).$$

Proof. The theorem follows from comparing the upper bound from Theorem 1.12 and the lower bound from $\|\chi_{\mathrm{en}}(KHI(Y,K))\|$. By [57, Lemma 3.2] and [24, Theorem 2.2] for L-space knots, and by [66, Section 4] for constrained knots, the upper bound matches the lower bound. ■

Remark 1.14. In the authors' previous work [42, Corollary 1.9], we proved Corollary 1.13 in the case where $H_1(Y(K);\mathbb{Z}) \cong \mathbb{Z}$. This is because the lower bound from $\|\chi_{\mathrm{gr}}(KHI(Y,K))\|$ may not equal to the upper bound in [43] when $H_1(Y\backslash \mathrm{int} N(K);\mathbb{Z})$ has torsions (cf. Example 1.4).

Remark 1.15. The result in Corollary 1.13 can be generalized to other $(1,1)$-knot K whose $\widehat{HFK}$ is totally determined by $\chi(\widehat{HFK}(Y,K))$, such as $(\pm 2,p,q)$ pretzel knots for odd integers p and q (cf. [23, Section 5]).

Any lens space Y has a standard Heegaard splitting of genus 1. A knot K in a lens space Y is called a **torus knot** if K can be isotoped to lie on the Heegaard torus of Y. This definition generalizes the usual torus knots in S^3. A knot K is called a **satellite knot** if $Y\backslash \mathrm{int} N(K)$ has an essential torus. A knot K is called **hyperbolic** if $Y\backslash K$ admits a hyperbolic metric of finite volume. By Thurston's Hyperbolization Theorem for Haken 3-manifolds, we have a good classification of knots in lens spaces.

Proposition 1.16 ([62, Proposition 3.1]). Suppose K is a knot in a lens space Y. If $Y(K)$ is irreducible, then K is either a torus knot, a satellite knot, or a hyperbolic knot.

It is straightforward to check that torus knots are $(1,1)$-knots, and their complements are Seifert fibred spaces.

Proposition 1.17 ([57, Theorem 5.1]). Knots in lens spaces with Seifert fibred complements are L-space knots.

Combining Proposition 1.17 and Corollary 1.13, we have the following result.

Corollary 1.18. For any torus knot K in a lens space Y, we have

$$\dim_{\mathbb{C}} KHI(Y,K) = \dim_{\mathbb{F}_2} \widehat{HFK}(Y,K).$$

Complements of many constrained knots are orientable hyperbolic 1-cusped manifolds with simple ideal triangulations. In particular, among 286 orientable 1-cusped manifolds that have ideal triangulations with at most five ideal tetrahedra, there are 232 manifolds that are complements of constrained knots. More examples can be found in [65]. Indeed, [66, Conjecture 2] conjectured that most constrained knots are hyperbolic knots.

1.1 Organization and sketch of the proofs

Suppose (M,γ) is a balanced sutured manifold. To sutured instanton and monopole homology together, we use $\underline{SHG}(M,\gamma)$ to denote both $\underline{SHI}(M,\gamma)$ and $\underline{SHM}(M,\gamma)$ (cf. [21, Section 2.1], see also [7]). It is a projectively transitive system, where each space in the system is isomorphic to $SHI(M,\gamma)$ (for $\underline{SHI}(M,\gamma)$) and $SHM(M,\gamma)$ (for $\underline{SHM}(M,\gamma)$). Note that it is different from the formal sutured homology $\mathbf{SHI}(M,\gamma)$ and $\mathbf{SHM}(M,\gamma)$ for instanton and monopole theory constructed in [42] because formal sutured homology corresponds to the untwisted theory in Baldwin and Sivek's construction [7], where $\underline{SHG}(M,\gamma)$ corresponds to the twisted theory in [7]. However, we have the equalities for graded Euler characteristics from [7, Theorem 7.7 and Theorem 9.21]

$$\chi_{\rm gr}(\underline{SHI}(M,\gamma)) = \chi_{\rm gr}(\mathbf{SHI}(M,\gamma)) \text{ and } \chi_{\rm gr}(\underline{SHM}(M,\gamma)) = \chi_{\rm gr}(\mathbf{SHM}(M,\gamma)).$$

Hence, the results for graded Euler characteristics of formal sutured homology in [42] can be applied to $\underline{SHG}(M,\gamma)$. In particular, we have

$$\chi_{\rm gr}(\underline{SHG}(M,\gamma)) = \chi_{\rm gr}(SFH(M,\gamma)). \tag{1.6}$$

Remark 1.19. In the previous version of this paper, we used the formal sutured homology to carry out proofs in the paper, but then noticed that some constructions might involve closures of balanced sutured manifolds of different genera. Moreover, the proof of the functoriality of contact gluing maps in [40] involves closures obtained from disconnected auxiliary surfaces, which can only be handled by a genus one version of Floer's excision theorem that is available in the twisted theory. Hence, in the current version, we use the twisted theory $\underline{SHG}(M,\gamma)$, which relates closures of different genera and closures with possibly disconnected auxiliary surfaces.

In Section 2, we review basic properties of $\underline{SHG}(M,\gamma)$, the gradings on $\underline{SHG}(M,\gamma)$ associated to admissible surfaces, and the maps on $\underline{SHG}(-M,-\gamma)$ associated to contact handle attachments (called **contact gluing maps**). Since we will use contact gluing maps frequently, it is more convenient to consider $(-M,-\gamma)$, the sutured manifold with the reverse orientation. Hence, we state all results with a minus sign.

In Section 3, we generalize the decomposition associated to a rationally null-homologous knot in [43, Section 4] to a connected rationally null-homologous tangle α, that is, $[\alpha]=0\in H_1(M,\partial M;\mathbb{Q})$. We write $M_\alpha = M\backslash {\rm int} N(\alpha)$. Then, by Lemma 3.19, we have

$$\mathrm{rk}_{\mathbb{Z}} H_1(M_\alpha;\mathbb{Z}) = \mathrm{rk}_{\mathbb{Z}} H_1(M;\mathbb{Z}) + 1,$$

and there is a surjective map

$$H_1(M_\alpha;\mathbb{Z}) \to H_1(M_\alpha;\mathbb{Z})/[m_\alpha] \cong H_1(M;\mathbb{Z}),$$

where m_α is the meridian of α. Moreover, after picking some suitable α, the pre-images of some torsions in $H_1(M;\mathbb{Z})$ are distinguished in the free part of $H_1(M_\alpha;\mathbb{Z})$. Since the difference in the free part can be detected by the gradings associated to admissible surfaces, we can decompose $\underline{SHG}(-M,-\gamma)$ by considering direct summands of $\underline{SHG}(-M_\alpha,-(\gamma\cup m_\alpha))$ in some gradings whose total dimension is the same as that of $\underline{SHG}(-M,-\gamma)$. The direct sum of these summands are denoted by $\mathcal{SHG}_\alpha(-M,-\gamma)$, which generalizes $\mathcal{I}_+(-\widehat{Y},\widehat{K})$ in [43, Section 4].

The above method can be applied iteratively for a tangle T with more than one component and finally we can distinguish all torsions in $H_1(M;\mathbb{Z})$ by choosing T such that $H_1(M_T;\mathbb{Z})$ is torsion-free (cf. Lemma 3.20). Similarly, we can identify $\underline{SHG}(-M,-\gamma)$ with a direct summand of $\underline{SHG}(-M_T,-(\gamma\cup m_T))$, where m_T is the union of meridians of tangle components of T. Since $H_1(M_T;\mathbb{Z})$ has no torsion, all torsions that are mixed on $H_1(M;\mathbb{Z})$ can be distinguished on $H_1(M_T;\mathbb{Z})$, and this provides the desired decomposition in Theorem 1.1.

Suppose j_* is the map on group rings induced by

$$j: H_1(M_T;\mathbb{Z})\to H_1(M;\mathbb{Z}).$$

Given the construction of $\mathcal{SHG}_T(-M,-\gamma)$, Equation (1.1) reduces to the following equation

$$\chi_{\rm en}(\underline{SHG}(-M,-\gamma)) := j_*(\chi_{\rm gr}(\mathcal{SHG}_T(-M,-\gamma))) = \chi(SFH(-M,-\gamma)). \tag{1.7}$$

We prove this equation by introducing a decomposition $\mathcal{SFH}_T(-M,-\gamma)$ of $SFH(-M,-\gamma)$ similar to $\mathcal{SHG}_T(-M,-\gamma)$. However, the construction of SFH is based on balanced diagrams of balanced sutured manifolds, which is different from the construction of $\underline{SHG}$ by closures. So we have to show that SFH satisfies the similar setups of $\underline{SHG}$ to construct $\mathcal{SFH}_T(-M,-\gamma)$. This is the main goal of Section 4, where we collect results for SFH parallel to $\underline{SHG}$, including gradings associated to admissible surfaces, the surgery exact triangle, the bypass exact triangle, and contact gluing maps.

Since $H_1(M_T;\mathbb{Z})$ is torsion-free, we can apply (1.6) to $(-M_T,-(\gamma\cup m_T))$ to obtain

$$\chi_{\rm gr}(\mathcal{SHG}_T(-M,-\gamma)) = \chi_{\rm gr}(\mathcal{SFH}_T(-M,-\gamma)) = \chi(\mathcal{SFH}_T(-M,-\gamma)). \tag{1.8}$$

By discussion on spin^c structures, we show

$$j_*(\chi(\mathcal{SFH}_T(-M,-\gamma))) = \chi(SFH(-M,-\gamma)). \tag{1.9}$$

Equations (1.8) and (1.9) imply Equation (1.7), which finishes the proof of Theorem 1.1. The detailed proof can be found in Section 5.

The proof of Theorem 1.3 is almost straightforward, based on the work of Lekili [39], and Baldwin and Sivek [11]. Since $\mathcal{SHG}_T(-M,-\gamma)$ is direct summands of $\underline{SHG}(-M_T,-\gamma\cup m_T)$ in some gradings, it suffices to prove the theorem when $H_1(M;\mathbb{Z})$ is torsion-free. In this case, the decomposition is just induced by admissible surfaces and Theorem 1.3 follows from the isomorphism

$$SHM(M,\gamma)\cong SFH(M,\gamma)\otimes\Lambda.$$

The detailed proof can be also found in Section 5.

In Section 6, we study knots whose $\dim_{\mathbb{C}} KHI$ are small and prove Theorem 1.5, Theorem 1.8, and analog theorems in Heegaard Floer theory.

1.2 Conventions

If it is not mentioned, all manifolds are smooth, oriented, and connected. Homology groups and cohomology groups are with $\mathbb{Z}$ coefficients, that is, $H_*(M):=H_*(M;\mathbb{Z})$ for any manifold M. For other coefficients (like $\mathbb{Q}$), we still write $H_*(M;\mathbb{Q})$. We write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$. For a simple closed curve on a surface, we do not distinguish between its homology class and itself. The algebraic intersection number of two curves α and β on a surface is denoted by $\alpha\cdot\beta$, while the number of intersection points between α and β is denoted by $|\alpha\cap\beta|$. A basis (m,l) of $H_1(T^2;\mathbb{Z})$ satisfies $m\cdot l=-1$. The **surgery** means the Dehn surgery and the slope q/p in the basis (m,l) corresponds to the curve $qm+pl$.

2 Twisted Sutured Homology

In this section, we collect useful properties of $\underline{SHG}$.

2.1 Notations, gradings, and Euler characteristics

Definition 2.1 ([29, 36]). A **balanced sutured manifold** (M,γ) consists of a compact oriented 3-manifold M with non-empty boundary together with a closed 1-submanifold γ on ∂M. Let $A(\gamma)=[-1,1]\times\gamma$ be an annular neighborhood of $\gamma\subset\partial M$, and let $R(\gamma)=\partial M\backslash\mathrm{int}(A(\gamma))$. They satisfy the following properties.

(1) Neither M nor $R(\gamma)$ has a closed component.

(2) If $\partial A(\gamma) = \partial R(\gamma)$ is oriented in the same way as γ, then we require this orientation of $\partial R(\gamma)$ induces one on $R(\gamma)$. The induced orientation on $R(\gamma)$ is called the **canonical orientation**.

(3) Let $R_+(\gamma)$ be the part of $R(\gamma)$ so that the canonical orientation coincides with the induced orientation on ∂M, and let $R_-(\gamma) = R(\gamma)\backslash R_+(\gamma)$. We require that $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$. If γ is clear in the contents, we simply write $R_\pm = R_\pm(\gamma)$.

The constructions of sutured instanton homology *SHI* and sutured monopole homology *SHM* for balanced sutured manifolds were originated by Kronheimer and Mrowka [36]. Later, Baldwin and Sivek [7] dealt with the naturality problem of these homologies and constructed $\underline{SHI}$ and $\underline{SHM}$. After that, several groups of people studies these homologies extensively, see for example [8, 12, 20, 41, 61].

In this paper, we will use $\underline{SHG}$ to denote both $\underline{SHI}$ and $\underline{SHM}$ and call it **twisted sutured homology**. The coefficient field is denoted by $\mathbb{F}$. For closed 3-manifolds and knots with basepoints, we can construct balanced sutured manifolds and then apply twisted sutured homology to them as follows.

Definition 2.2. Suppose that Y is a closed 3-manifold and $z \in Y$ is a basepoint. Let $Y(1)$ be obtained from Y by removing a 3-ball containing z, and let δ be a simple closed curve on $\partial Y(1) \cong S^2$. Suppose that $K \subset Y$ is a knot and w is a basepoint on K. Let $Y(K)$ be the knot complement of K, and let $\gamma = m \cup (-m)$ consist of two meridians with opposite orientations of K near w. Then $(Y(1), \delta)$ and $(Y(K), \gamma)$ are balanced sutured manifolds. Define

$$\underline{HG}(Y,z) := \underline{SHG}(Y(1),\delta) \text{ and } \underline{KHG}(Y,K,w) := \underline{SHG}(Y(K),\gamma).$$

Convention. Different choices of the basepoints give isomorphic vector spaces. Since in the current paper we only care about the isomorphism class of the vector spaces, we omit the basepoints and simply write $\underline{HG}(Y)$ and $\underline{KHG}(Y,K)$ instead.

If S is a properly embedded surface in M with some admissible conditions. The first author [41] constructed a grading on $\underline{SHG}(M,\gamma)$ (with some pioneering work done by Kronheimer and Mrowka [35] and Baldwin and Sivek [12]).

Definition 2.3 ([20]). Suppose (M,γ) is a balanced sutured manifold and $S \subset M$ is a properly embedded surface. The surface S is called an **admissible surface** if the followings hold.

(1) Every boundary component of S intersects γ transversely and nontrivially.
(2) $\frac{1}{2}|S \cap \gamma| - \chi(S)$ is an even integer.

Theorem 2.4 ([40, 41]). Suppose (M, γ) is a balanced sutured manifold and $S \subset (M, \gamma)$ is an admissible surface. Then there is a $\mathbb{Z}$-grading on $\underline{SHG}(M, \gamma)$ induced by S, which we write as

$$\underline{SHG}(M, \gamma) = \bigoplus_{i \in \mathbb{Z}} \underline{SHG}(M, \gamma, S, i).$$

This decomposition satisfies the following properties.

(1) Suppose $n = \frac{1}{2}|\partial S \cap \gamma|$. If $|i| > \frac{1}{2}(n - \chi(S))$, then $\underline{SHG}(M, \gamma, S, i) = 0$.
(2) If there is a sutured manifold decomposition $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ in the sense of Gabai [19], then we have

$$\underline{SHG}(M, \gamma, S, \frac{1}{2}(n - \chi(S))) \cong \underline{SHG}(M', \gamma').$$

(3) For any $i \in \mathbb{Z}$, we have

$$\underline{SHG}(M, \gamma, S, i) = \underline{SHG}(M, \gamma, -S, -i).$$

(4) For any $i \in \mathbb{Z}$, we have

$$\underline{SHG}(M, -\gamma, S, i) \cong \underline{SHG}(M, \gamma, S, -i).$$

(5) For any $i \in \mathbb{Z}$, we have

$$\underline{SHG}(-M, \gamma, S, i) \cong \mathrm{Hom}_{\mathbb{F}}(\underline{SHG}(M, \gamma, S, -i), \mathbb{F}).$$

If $S \subset (M, \gamma)$ is not admissible, then we can perform an isotopy on S to make it admissible.

Definition 2.5. Suppose (M, γ) is a balanced sutured manifold, and S is a properly embedded surface. A **stabilization** of S is a surface S' obtained from S by isotopy in the following sense. This isotopy creates a new pair of intersection points:

$$\partial S' \cap \gamma = (\partial S \cap \gamma) \cup \{p_+, p_-\}.$$

We require that there are arcs $\alpha \subset \partial S'$ and $\beta \subset \gamma$, oriented in the same way as $\partial S'$ and γ, respectively, and the following hold.

(1) $\partial \alpha = \partial \beta = \{p_+, p_-\}$.

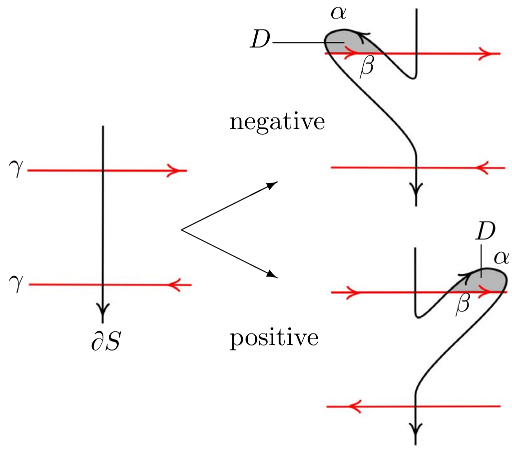

Fig. 1. The positive and negative stabilizations of S.

(2) α and β cobound a disk D with $\mathrm{int}(D) \cap (\gamma \cup \partial S') = \emptyset$.

The stabilization is called **negative** if ∂D is the union of α and β as an oriented curve. It is called **positive** if $\partial D = (-\alpha) \cup \beta$. See Figure 1. We denote by $S^{\pm k}$ the surface obtained from S by performing k positive or negative stabilizations, respectively.

Remark 2.6. The definition of stabilizations of a surface depends on the orientations of the suture and the surface. If we reverse the orientation of the suture or the surface, then positive and negative stabilizations switch between each other.

One can also relate the gradings associated to different stabilizations of a fixed surface.

Theorem 2.7 ([41, Proposition 4.3] and [61, Proposition 4.17]). Suppose (M, γ) is a balanced sutured manifold and S is a properly embedded surface in M, which intersects the suture γ transversely. Suppose S has a distinguished boundary component so that all the stabilizations mentioned below are performed on this boundary component. Then, for any $p, k, l \in \mathbb{Z}$ so that the stabilized surfaces S^p and S^{p+2k} are both admissible, we have

$$\underline{SHG}(M, \gamma, S^p, l) = \underline{SHG}(M, \gamma, S^{p+2k}, l+k).$$

Note S^p is a stabilization of S as introduced in Definition 2.5, and, in particular, $S^0 = S$.

If we have multiple admissible surfaces, then they together induce a multi-grading.

Theorem 2.8 ([20, Proposition 1.14]). Suppose (M,γ) is a balanced sutured manifold and $S_1,\dots,S_n$ are admissible surfaces in (M,γ). Then there exists a $\mathbb{Z}^n$-grading on $\underline{SHG}(M,\gamma)$ induced by $S_1,\dots,S_n$, which we write as

$$\underline{SHG}(M,\gamma)=\bigoplus_{(i_1,\dots,i_n)\in\mathbb{Z}^n}\underline{SHG}(M,\gamma,(S_1,\dots,S_n),(i_1,\dots,i_n)).$$

Theorem 2.9 ([20, Theorem 1.12]). Suppose (M,γ) is a balanced sutured manifold and $\alpha\in H_2(M,\partial M)$ is a nontrivial homology class. Suppose S_1 and S_2 are two admissible surfaces in (M,γ) such that

$$[S_1]=[S_2]=\alpha\in H_2(M,\partial M).$$

Then, there exists a constant C so that

$$\underline{SHG}(M,\gamma,S_1,l)=\underline{SHG}(M,\gamma,S_2,l+C).$$

Based on the $\mathbb{Z}^n$ grading from Theorem 2.8, we can define the graded Euler characteristic.

Definition 2.10. Suppose (M,γ) is a balanced sutured manifold and $S_1,\dots,S_n$ are admissible surfaces in (M,γ) such that $[S_1],\dots,[S_n]$ generate $H_2(M,\partial M)$. Let $\rho_1,\dots,\rho_n\in H'=H_1(M)/\mathrm{Tors}$ satisfying $\rho_i\cdot S_j=\delta_{i,j}$. The **graded Euler characteristic** of $\underline{SHG}(M,\gamma)$ is

$$\chi_{\mathrm{gr}}(\underline{SHG}(M,\gamma)):=\sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n}\chi(\underline{SHG}(M,\gamma,(S_1,\dots,S_n),(i_1,\dots,i_n)))\cdot(\rho_1^{i_1}\cdots\rho_n^{i_n})\in\mathbb{Z}[H']/\pm H'.$$

Remark 2.11. By Theorem 2.9, the definition of graded Euler characteristic is independent of the choices of $S_1,\dots,S_n$ if we regard it as an element in $\mathbb{Z}[H']/\pm H'$. If the admissible surfaces $S_1,\dots,S_n$ and a particular closure of (M,γ) is fixed, then the ambiguity of $\pm H'$ can be removed.

2.2 Contact handles and bypasses

Suppose $(M,\gamma)\subset(M',\gamma')$ is a proper inclusion of balanced sutured manifolds and suppose ξ is a contact structure on $M'\backslash\mathrm{int}M$ with dividing sets $\gamma'\cup(-\gamma)$. For monopole theory and instanton theory, Baldwin and Sivek [8, 9] (see also [40]) constructed a **contact**

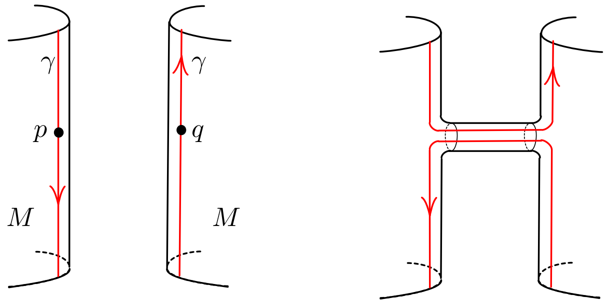

Fig. 2. Left, the sutured manifold (M, γ) with two points p and q on the suture. Right, the 1-handle attachment along p and q.

gluing map

$$\Phi_\xi : \underline{SHG}(-M, -\gamma) \to \underline{SHG}(-M', -\gamma')$$

based on contact handle decompositions and the first author [40] showed that the map is functorial, that is, it is independent of the contact handle decompositions and gluing two contact structures induces composite maps. In this subsection, we will describe the maps associated to contact 1- and 2-handle attachments, and bypass attachments (cf. [26]).

Contact 1-handle. Suppose D_- and D_+ are disjoint embedded disks in ∂M, which each intersect γ in a single properly embedded arc. Consider the standard contact structure ξ_{std} on the 3-ball B^3. We glue $(D^2 \times [-1, 1], \xi_{D^2}) \cong (B^3, \xi_{\text{std}})$ to (M, γ) by diffeomorphisms

$$D^2 \times \{-1\} \to D_- \text{ and } D^2 \times \{+1\} \to D_+,$$

which preserve and reverse orientations, respectively, and identify the dividing sets with the sutures. Then we round corners as shown in Figure 2 (cf. [9, Figure 2]). Let (M_1, γ_1) be the resulting sutured manifold.

Suppose (Y, R) is a closure of (M_1, γ_1). By [9, Section 3.2], it is also a closure of (M, γ). Define the map associated to the contact 1-handle attachment by the identity map

$$C_{h^1} = C_{h^1, D_-, D_+} := \text{id} : \underline{SHG}(-M, -\gamma) \xrightarrow{=} \underline{SHG}(-M_1, -\gamma_1).$$

Contact 2-handle. Suppose μ is an embedded curve in ∂M, which intersects γ in two points. Let $A(\mu)$ be an annular neighborhood of λ intersecting γ in two cocores. We

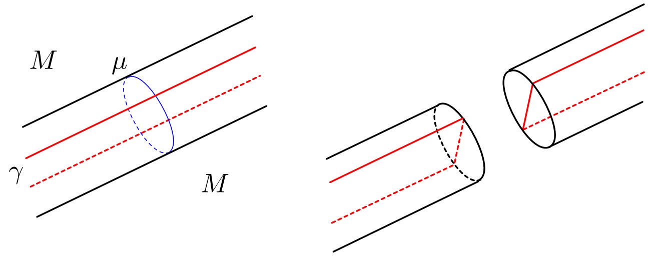

Fig. 3. Left, the sutured manifold (M,γ) and the curve $\beta \subset \partial M$ that intersects γ at two points. Right, the 2-handle attachment along the curve μ.

glue $(D^2 \times [-1,1], \xi_{D^2}) \cong (B^3, \xi_{\mathrm{std}})$ to (M,γ) by an orientation-reversing diffeomorphism

$$\partial D^2 \times [-1,1] \to A(\mu),$$

which identifies positive regions with negative regions. Then we round corners as shown in Figure 3 (cf. [9, Figure 3]). Let (M_2,γ_2) be the resulting sutured manifold.

We construct the map associated to the contact 2-handle attachment as follows. Let μ' be the knot obtained by pushing μ into M slightly. Suppose (N,γ_N) is the manifold obtained from (M,γ) by a 0-surgery along μ' with respect to the framing from ∂N. By [9, Section 3.3], the sutured manifold (N,γ_N) can be obtained from (M_2,γ_2) by attaching a contact 1-handle. Since $\mu' \subset \mathrm{int}(M)$, the construction of the closure of (M,γ) does not affect μ'. Thus, we can construct a cobordism between closures of (M,γ) and (N,γ_N) by attaching a 4-dimensional 2-handle associated to the surgery on μ'. This cobordism induces a cobordism map

$$C_{\mu'} : \underline{SHG}(-M,-\gamma) \to \underline{SHG}(-N,-\gamma_N).$$

Consider the identity map

$$\iota : \underline{SHG}(-M_2,-\gamma_2) \xrightarrow{=} \underline{SHG}(-N,-\gamma_N).$$

Define the the map associated to the contact 2-handle attachment as

$$C_{h^2} = C_{h^2,\mu} := \iota^{-1} \circ C_{\mu'} : \underline{SHG}(-M,-\gamma) \to \underline{SHG}(-M_2,-\gamma_2).$$

Bypass attachment. Suppose α is an embedded arc in ∂M, which intersects γ in three points. Let D be a disk neighborhood of α intersecting γ in three arcs. There are

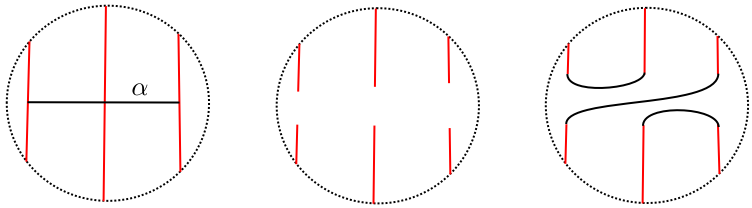

Fig. 4. The bypass arc and the bypass attachment, where the orientation of ∂M is pointing out.

six endpoints after cutting γ along α. We replace three arcs in D with another three arcs as shown in Figure 4. Let (M,γ') be the resulting sutured manifold. The arc α is called a **bypass arc** and this procedure is called **bypass attachment** along α.

By Ozbagci [47, Section 3], the bypass attachment can be recovered by contact handle attachments as follows. First, one can attach a contact 1-handle along two endpoints of α. Then one can attach a contact 2-handle along a circle that is the union of α and an arc on the attached 1-handle. Topologically, the 1-handle and the 2-handle form a canceling pair, so the diffeomorphism type of the 3-manifold does not change. However, the contact structure is changed, and the suture γ is replaced by γ'. We define the **bypass map** associated to the bypass attachment as

$$\psi_\alpha := C_{h^2}\circ C_{h^1} : \underline{SHG}(-M,-\gamma)\to \underline{SHG}(-M,-\gamma').$$

In [8, 12], Baldwin and Sivek proved the bypass exact triangle for sutured monopole Floer homology and sutured instanton Floer homology.

Theorem 2.12 ([8, Theorem 5.2] and [12, Theorem 1.21]). Suppose (M,γ_1), (M,γ_2), (M,γ_3) are balanced sutured manifolds such that the underlying 3-manifolds are the same, and the sutures γ_1, γ_2, and γ_3 only differ in a disk shown in Figure 5. Then there exists an exact triangle

$$\begin{tikzcd}
\underline{SHG}(-M,-\gamma_1) \arrow[rr, "\psi_1"] & & \underline{SHG}(-M,-\gamma_2) \arrow[dl, "\psi_2"] \\
& \underline{SHG}(-M,-\gamma_3) \arrow[ul, "\psi_3"] &
\end{tikzcd} \tag{2.1}$$

Moreover, the maps ψ_i are induced by cobordisms, hence is homogeneous with respect to the relative $\mathbb{Z}_2$ grading on $\underline{SHG}(M,\gamma_i)$.

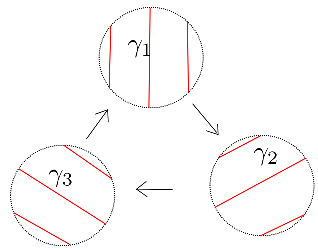

Fig. 5. The bypass triangle.

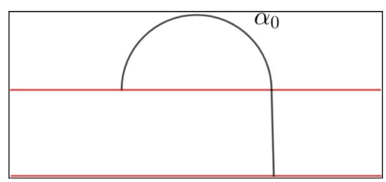

Fig. 6. A trivial bypass.

The following proposition is straightforward from the description of the bypass map.

Proposition 2.13. Suppose (M,γ) is a balanced sutured manifold and $S\subset(M,\gamma)$ is an admissible surface. Suppose the disk as in Figure 5, where we perform the bypass change, is disjoint from ∂S. Let γ_2 and γ_3 be the resulting two sutures. Then all the maps in the bypass exact triangle (2.1) are grading preserving, that is, for any $i\in\mathbb{Z}$, we have an exact triangle

$$\begin{tikzcd}
\underline{SHG}(-M,-\gamma_1,S,i) \arrow[rr, "\psi_{1,i}"] & & \underline{SHG}(-M,-\gamma_2,S,i) \arrow[dl, "\psi_{2,i}"] \\
& \underline{SHG}(-M,-\gamma_3,S,i) \arrow[ul, "\psi_{3,i}"] &
\end{tikzcd}$$

where $\psi_{k,i}$ are the restriction of ψ_k in (2.1).

A special bypass arc α_0 is depicted in Figure 6, where the bypass attachment along α is called a **trivial bypass** (cf. [27, Section 2.3]). Attaching a trivial bypass does not change the suture on ∂M and induces a product contact structure on $\partial M\times I$. The functoriality of the contact gluing maps indicates the following proposition.

Proposition 2.14 (). A trivial bypass on (M,γ) induces an identity map on $\underline{SHG}(M,\gamma)$.

3 Decomposition Associated to Tangles

Suppose K is a rationally null-homologous knot in a closed 3-manifold Y, that is, $[K]=0\in H_1(Y;\mathbb{Q})$. Suppose q is the order of $[K]$, that is, q is the smallest number satisfying $q[K]=0\in H_1(Y;\mathbb{Z})$. In [43, Section 4], we construct a decomposition

$$I^{\sharp}(Y)\cong\bigoplus_{i=0}^{q}I^{\sharp}(Y,i).$$

This decomposition provides a candidate for the counterpart of the torsion spin^c decompositions in monopole theory and Heegaard Floer theory.

In this section, we generalize this decomposition to rationally null-homologous tangles in balanced sutured manifolds. There is no essential difference between the proofs for knots and tangles. All arguments apply to both sutured instanton and monopole homology, so we can safely use $\underline{SHG}$.

3.1 Basic setups

In this subsection, we review the construction for tangles and collect important lemmas in [43, Section 3.2], with mild modifications.

Suppose (M,γ) is a balanced sutured manifold. Suppose $T=T_1\cup\cdots\cup T_m$ is a **vertical tangle** in (M,γ) (cf. [63, Definition 1.1]), that is, a properly embedded 1-submanifold with

$$|T_i\cap R_+(\gamma)|=|T_i\cap R_-(\gamma)|=1.$$

Let T_i be oriented from $R_+(\gamma)$ to $R_-(\gamma)$. Throughout this subsection, we consider one component α of T and assume it is rationally null-homologous, that is, $[\alpha]=0\in H_1(M,\partial M;\mathbb{Q})$. Without loss of generality, suppose $\alpha=T_1$.

We can construct a new balanced sutured manifold (M_T,γ_T) as follows. Let M_T be obtained from M by removing a neighborhood $N(T)=\bigcup_{i=1}^{m}N(T_i)$ of T. Suppose γ_i is a positively oriented meridian of T_i on $\partial N(T_i)$. Define

$$\gamma_T=\gamma\cup\gamma_1\cup\cdots\cup\gamma_m.$$

Since α is rationally null-homologous, there exists a surface S in M with ∂S consisting of arcs $\beta_1,\ldots,\beta_k$ and q copies of α for some integers k and q. Here q is the order of α.

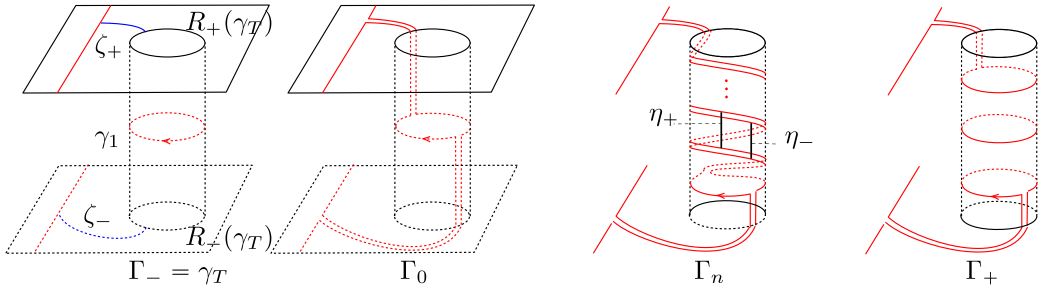

Fig. 7. The arcs ζ_+, ζ_-, the sutures $\Gamma_-, \Gamma_0, \Gamma_n, \Gamma_+$, and the bypass arcs η_+, η_-.

The surface S can be modified into a properly embedded surface S_T in M_T as follows. First, for q arcs in ∂S parallel to α, we isotope them to be on $\partial N(\alpha)$. Then $\beta_1, \dots, \beta_k$ can be regarded as arcs on ∂M_T. Second, We can isotope S to make it intersect $T_2, \dots, T_m$ transversely. Then removing disks in $N(T_i) \cap S$ for all $i = 2, \dots, m$ induces a properly embedded surface S_T in M_T. Note that ∂S_T intersects γ_1 at q points, one for each arc parallel to α, and the part of ∂S_T on $\partial N(T_i)$ consists of circles parallel to γ_i for $i = 2, \dots, m$.

Suppose p_+ and p_- are the endpoints of α on $R_+(\gamma)$ and $R_-(\gamma)$, respectively. Choose an arc $\zeta_+ \subset R_+(\gamma)$ connecting p_+ and γ. The arc ζ_+ induces an arc on $R_+(\gamma_T)$ connecting γ_1 to γ such that the part on $\partial N(\alpha)$ is parallel to α. We still denote this arc by ζ_+ for simplicity. Similarly, we can choose an arc $\zeta_- \subset R_-(\gamma_T)$ connecting γ_1 to γ.

Let Γ_0 be obtained from γ_T by band sum operations along ζ_+ and ζ_-. Then let Γ_n be obtained from Γ_0 by twisting along $(-\gamma_1)$ for n times. Moreover, let Γ_+ be the suture as depicted in Figure 7 and let $\Gamma_- = \gamma_T$.

Remark 3.1. The construction of ζ_+ and ζ_- here is a little different from the one in [43, Section 3.2], where we used β_1 to construct $\zeta_\pm$ and removed a trivial tangle from M_T to obtain a manifold M_{T_0}. Hence, the construction of $\Gamma_n, \Gamma_\pm$ is also different. In particular, they were on M_{T_0} in the construction of [43, Section 3.2]. However, it turns out that removing the trivial tangle is not necessary and we can decompose M_{T_0} along a product disk to recover M_T in [43, Section 3.2, Step 3]. Thus, we can consider sutures on M_T and all results in [43, Section 3.2] apply without essential change. Also, the conditions that $\zeta_\pm$ are disjoint from $\beta_1, \dots, \beta_k$ are not essential.

There are two straightforward choices of bypass arcs on Γ_n in Figure 7, denoted by η_+ and η_-, respectively. It is straightforward to check that these two bypass

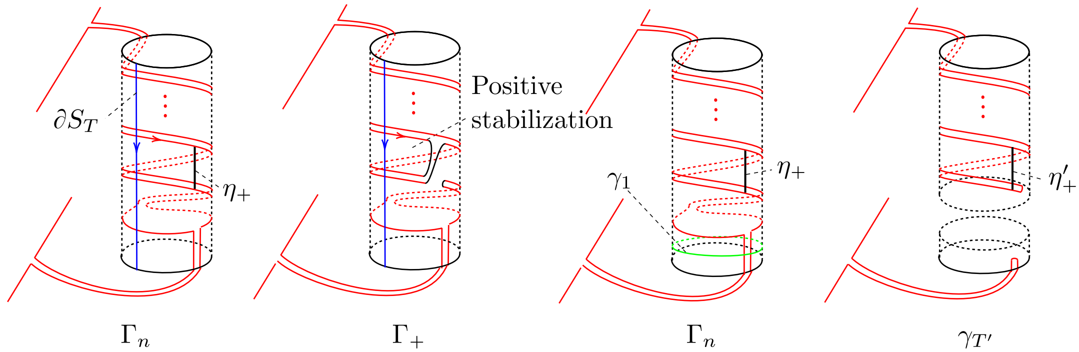

Fig. 8. Left two subfigures, the bypass attachment along η_+. Right two subfigures, the bypass arcs before and after the contact 2-handle attachment.

arcs induce the following bypass exact triangles from Theorem 2.12 (cf. the left two subfigures of Figure 8).

$$\begin{tikzcd}
\underline{SHG}(-M_T, -\Gamma_{n-1}) \arrow[rr, "\psi^{n-1}_{\pm,n}"] & & \underline{SHG}(-M_T, -\Gamma_n) \arrow[dl, "\psi^{n}_{\pm,\pm}"] \\
& \underline{SHG}(-M_T, -\Gamma_\pm) \arrow[ul, "\psi^{\pm}_{\pm,n-1}"] &
\end{tikzcd} \tag{3.1}$$

The bypasses are attached along η_+ and η_- from the exterior of the 3-manifold M_{T_0}, though the point of view in Figure 7 is from the interior of the manifold. So readers have to take extra care when performing these bypass attachments.

Since the bypass arcs η_+ and η_- are disjoint from ∂S_T, the bypass maps in the exact triangles (3.1) preserve gradings associated to S_T by Proposition 2.13. We describe it precisely as follows.

Definition 3.2. Suppose (M, γ) is a balanced sutured manifold and S is an admissible surface in (M, γ). For any $i, j \in \mathbb{Z}$, define

$$\underline{SHG}(M, \gamma, S, i)[j] = \underline{SHG}(M, \gamma, S, i - j).$$

Lemma 3.3 ([43, Lemma 3.15 and Lemma 3.18]). For any $j \in \mathbb{N} \cup \{+, -\}$, there exists admissible surfaces S_j with respect to (M_T, Γ_j) obtained from S_T by stabilizations and integers i^j_{max} and i^j_{min} such that

$$\lim_{n\to+\infty} i^n_{max} = +\infty, \lim_{n\to+\infty} i^n_{min} = -\infty,$$

and

$$\underline{SHG}(-M_T, -\Gamma_j, S_j, i) = 0 \text{ for } i \notin [i^j_{min}, i^j_{max}].$$

Moreover, for any $n \in \mathbb{N}$, there are two exact triangles

$$\begin{tikzcd}
\underline{SHG}(-M_T, -\Gamma_n, S_n)[i^{n+1}_{min} - i^n_{min}] \arrow[r, "\psi^n_{+,n+1}"] & \underline{SHG}(-M_T, -\Gamma_{n+1}, S_{n+1}) \arrow[dl, "\psi^{n+1}_{+,+}"] \\
\underline{SHG}(-M_T, -\Gamma_+, S_+)[i^{n+1}_{max} - i^+_{max}] \arrow[u, "\psi^+_{+,n}"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\underline{SHG}(-M_T, -\Gamma_n, S_n)[i^{n+1}_{max} - i^n_{max}] \arrow[r, "\psi^n_{-,n+1}"] & \underline{SHG}(-M_T, -\Gamma_{n+1}, S_{n+1}) \arrow[dl, "\psi^{n+1}_{-,-}"] \\
\underline{SHG}(-M_T, -\Gamma_-, S_-)[i^{n+1}_{min} - i^-_{min}] \arrow[u, "\psi^-_{-,n}"] &
\end{tikzcd} .$$

Furthermore, all maps in the above two exact triangles are grading preserving.

Remark 3.4. For $j \in \mathbb{N} \cup \{+,-\}$, the surfaces S_j and integers i^j_{max}, i^j_{min} were defined explicitly in [43, Step 2 in Section 3.2] by some stabilizations. However, three conditions about S_T at the start of [43, Step 2 in Section 3.2] are not necessary. We can choose S_j to be either S_T or S^{-1}_T (the negative stabilization of S_T with respect to Γ_j, cf. Definition 2.5), which is admissible with respect to Γ_j. The choice is denoted by $S_j = S^{\tau(j)}_T$ for $\tau(j) \in \{0, -1\}$. Explicitly, $\tau(j) = 0$ if S_j is admissible and $\tau(j) = -1$ if S_j is not. For the definitions of i^j_{max}, i^j_{min}, consider the closure (Y_j, R_j) of (M_T, Γ_j) such that S_j extends to a closed surface $\bar{S}_j \subset Y_j$. Define

$$i^j_{max} = -\frac{1}{2}\chi(\bar{S}_j), \text{ and } i^j_{min} = \frac{1}{2}\chi(\bar{S}_j) - \tau(j).$$

Moreover, we have

$$\chi(\bar{S}_j) = \chi(S_j) - \frac{1}{2}|S_j \cap \Gamma_j|$$

Hence

$$\chi(\bar{S}_-) = \chi(\bar{S}_+) - q + \tau(-) \text{ and } \chi(\bar{S}_n) = \chi(\bar{S}_+) - nq + \tau(n) \text{ for } n \in \mathbb{N},$$

where q is the order of the tangle α. Since the surfaces after stabilizations are disjoint from the bypass arcs, the bypass maps are grading preserving by Proposition 2.13. The vanishing results follow from term (2) of Theorem 2.4, and a priori we do not know if $\underline{SHG}(-M_T, -\Gamma_j, S_j, i)$ is non-vanishing for $i \in \{i^j_{min}, i^j_{min}\}$.

From the vanishing results and the exact triangles in Lemma 3.3, the following lemma is straightforward. For any $i \in \mathbb{Z}, n \in \mathbb{N}$, let $\psi^{n,i}_{\pm,n+1}$ be the restriction of $\psi^n_{\pm,n+1}$ on the i-th grading associated to S_n.

Lemma 3.5 ([43, Lemma 3.20]). The map

$$\psi^{n,i}_{+,n+1} : \underline{SHG}(-M_T, -\Gamma_n, S_n, i) \to \underline{SHG}(-M_T, -\Gamma_{n+1}, S_{n+1}, i - (i^n_{min} - i^{n+1}_{min}))$$

is an isomorphism if $i < P_n := i^n_{min} + (n+1)q - \tau(+)$. Similarly, the map

$$\psi^{n,i}_{-,n+1} : \underline{SHG}(-M_T, -\Gamma_n, S_n, i) \to \underline{SHG}(-M_T, -\Gamma_{n+1}, S_{n+1}, i + (i^{n+1}_{max} - i^n_{max}))$$

is an isomorphism if $i > \rho_n := i^n_{max} - nq$.

Proof. Note that

$$\begin{aligned} i^{n+1}_{max} + (i^n_{min} - i^{n+1}_{min}) - (i^+_{max} - i^+_{min}) &= i^n_{min} + (i^{n+1}_{max} - i^{n+1}_{min}) - (i^+_{max} - i^+_{min}) \\ &= i^n_{min} + (-\chi(\bar{S}_+) + (n+1)q) - (-\chi(\bar{S}_+) + \tau(+)) \\ &= i^n_{min} + (n+1)q - \tau(+). \end{aligned}$$

$$\begin{aligned} i^{n+1}_{min} - (i^{n+1}_{max} - i^n_{max}) + (i^-_{max} - i^-_{min}) &= i^n_{max} - (i^{n+1}_{max} - i^{n+1}_{min}) + (i^-_{max} - i^-_{min}) \\ &= i^n_{max} - (-\chi(\bar{S}_+) + (n+1)q) + (-\chi(\bar{S}_+) - q) \\ &= i^n_{max} - nq. \end{aligned}$$

■

There is another important exact triangle induced by the surgery exact triangle.

Lemma 3.6 ([43, Lemma 3.21]). Suppose $T' = T\backslash\alpha = T_2 \cup \cdots \cup T_m$. Then for any $n \in \mathbb{N}$, there is an exact triangle

$$\begin{tikzcd}
\underline{SHG}(-M_T, -\Gamma_n) \arrow[rr] & & \underline{SHG}(-M_T, -\Gamma_{n+1}) \arrow[dl, "F_{n+1}"] \\
& \underline{SHG}(-M_{T'}, -\gamma_{T'}) \arrow[ul, "G_n"] &
\end{tikzcd} \tag{3.2}$$

Furthermore, we have two commutative diagrams related to $\psi^n_{+,n+1}$ and $\psi^n_{-,n+1}$, respectively

$$\begin{tikzcd}
\underline{SHG}(-M_T, -\Gamma_n) \arrow[rr, "\psi^n_{\pm,n+1}"] & & \underline{SHG}(-M_T, -\Gamma_{n+1}) \\
& \underline{SHG}(-M_{T'}, -\gamma_{T'}) \arrow[ul, "G_n"] \arrow[ur, "G_{n+1}"'] &
\end{tikzcd}$$

The proof of this lemma is used in Section 4. So we sketch the proof for the reader's convenience.

Proof. Sketch of the proof of Lemma 3.6 Recall that γ_1 is the meridian of α on ∂M_T. Let γ_1' be the curve obtained by pushing γ_1 into the interior of M_T, with the framing induced by ∂M_T. The exact triangle in (3.2) comes from the surgery exact triangle along γ_1: 1-surgery twists the suture and leads to Γ_{n+1}; 0-surgery corresponds to a 2-handle attachment along γ_1 and hence fills the tangle that leads to the sutured manifold $(M_{T'}, \gamma_{T'})$. The commutativity coming from the fact that the surgery curve γ_1' and the bypass arc η_+' are disjoint from each other, and the fact that the corresponding bypass after the 0-surgery along γ_1' is trivial. ∎

Remark 3.7. Indeed, we have two more commutative diagrams about F_n:

$$\begin{tikzcd}
\underline{SHG}(-M_T, -\Gamma_n) \arrow[rr, "\psi^n_{\pm,n+1}"] \arrow[dr, "F_n"] & & \underline{SHG}(-M_T, -\Gamma_{n+1}) \arrow[dl, "F_{n+1}"'] \\
& \underline{SHG}(-M_{T'}, -\gamma_{T'}) &
\end{tikzcd}$$

The proofs are also similar.

Combining Lemma 3.3, Lemma 3.5, and Lemma 3.6, we get the following result.

Lemma 3.8 ([43, Lemma 3.22]). For a large enough integer n, the map G_n in Lemma 3.6 is zero. Hence, F_{n+1} is surjective by the exact triangle (3.2).

The map F_{n+1} in Lemma 3.6 is a map associated to a contact 2-handle attachment. We have the following grading preserving result.

Lemma 3.9 ([42, Section 4.2]). Suppose (M,γ) is a balanced sutured manifold and $S \subset (M,\gamma)$ is an admissible surface. Suppose $\alpha \subset M$ is a properly embedded arc that intersects S transversely and $\partial\alpha \cap \partial S = \emptyset$. Let $N = M \backslash \mathrm{int} N(\alpha), S_N = S \cap N$, and let $\mu \subset \partial N$ be the meridian of α that is disjoint from ∂S_N. Suppose γ_N is a suture on ∂N such that (N,γ_N) is a balanced sutured manifold and attaching a contact 2-handle along μ gives (M,γ). Let $C_{h^2,\mu}$ be the map associated to the contact 2-handle attachment. Then for any $i \in \mathbb{Z}$, we have

$$C_{h^2,\mu}(\underline{SHG}(-N,-\gamma_N,S_N,i)) \subset \underline{SHG}(-M,-\gamma,S,i).$$

3.2 One tangle component

In this subsection, we apply lemmas in Section 3.1 to obtain a decomposition of twisted sutured homology associated to one tangle component. The results in this subsection are a generalization of [43, Section 4.3], where we dealt with rationally null-homologous knots. The proofs are almost identical, so we omit details and only point out the difference.

We adapt the notations in Subsection 3.1. Suppose (M,γ) is a balanced sutured manifold and suppose $T \subset (M,\gamma)$ is a vertical tangle with only one component $\alpha = T_1$, which is rationally null-homologous of order q. Let M_T be the manifold obtained from M by removing a neighborhood of T and let $\gamma_T = \gamma \cup m_\alpha$, where m_α is a positively oriented meridian of α.

We start with the following lemma, which roughly says the summands in the "middle" gradings of $\underline{SHG}(-M_T,-\Gamma_n)$ associated to S_n are cyclic of order q.

Lemma 3.10. Suppose $n \in \mathbb{N}$ and $i_1, i_2 \in \mathbb{Z}$ satisfying $i_1, i_2 \in (\rho_n, P_n)$ and $i_1 - i_2 = q$, where ρ_n and P_n are constants in Lemma 3.5:

$$\rho_n = i^n_{max} - nq \text{ and } P_n = i^n_{min} + (n+1)q - \tau(+).$$

Then we have

$$\underline{SHG}(-M_T, -\Gamma_n, S_n, i_1) \cong \underline{SHG}(-M_T, -\Gamma_n, S_n, i_2).$$

Proof. Based on Lemma 3.5, the proof is similar to that of [43, Lemma 4.20]. Here we only include some key steps as follows.

Since $i_1 < P_n$, by Lemma 3.5, we know

$$\underline{SHG}(-M_T, -\Gamma_n, S_n, i_1) \cong \underline{SHG}(-M_T, -\Gamma_{n+1}, S_n, i_1 - (i^n_{min} - i^{n+1}_{min})).$$

Similarly, since $i_2 > \rho_n$, we know that

$$\underline{SHG}(-M_T, -\Gamma_n, S_n, i_2) \cong \underline{SHG}(-M_T, -\Gamma_{n+1}, S_n, i_2 + (i^{n+1}_{max} - i^n_{max})).$$

By definitions of i^n_{min} and i^n_{max} in Remark 3.4, we have

$$\begin{aligned} i_1 - i^n_{min} + i^{n+1}_{min} &= i_2 - i^n_{max} + i^{n+1}_{max} + q + (i^n_{max} - i^n_{min}) - (i^{n+1}_{max} - i^{n+1}_{min}) \\ &= i_2 - i^n_{max} + i^{n+1}_{max} + q + (-\chi(\bar{S}_n) - \tau(n)) - (-\chi(\bar{S}_{n+1}) - \tau(n+1)) \\ &= i_2 - i^n_{max} + i^{n+1}_{max} + q + (-\chi(\bar{S}_+) + nq) - (-\chi(\bar{S}_+) + (n+1)q) \\ &= i_2 - i^n_{max} + i^{n+1}_{max}. \end{aligned}$$

Hence, we obtain the desired result. ■

Note that

$$\begin{aligned} P_n - \rho_n &= (i^n_{min} + (n+1)q - \tau(+)) - (i^n_{max} - nq) \\ &= -(i^n_{max} - i^n_{min}) - \tau(+) + (2n+1)q \\ &= -(-\chi(\bar{S}_+) + nq) - \tau(+) + (2n+1)q \\ &= \chi(\bar{S}_+) - \tau(+) + (n+1)q. \end{aligned}$$

Thus, the difference of P_n and ρ_n can be infinitely large.

Definition 3.11. Define $Q_n = P_n - q + \tau(+)$. Suppose $n \in \mathbb{N}$ satisfies $Q_n - \rho_n > q$, define

$$\mathcal{SHG}_\alpha(-M, -\gamma, i) := \underline{SHG}(-M_T, -\Gamma_n, S_n, Q_n - i),$$

and

$$\mathcal{SHG}_\alpha(-M,-\gamma) := \bigoplus_{i=1}^{q} \mathcal{SHG}_\alpha(-M,-\gamma,i).$$

Remark 3.12. From the definitions of Q_n, P_n, ρ_n, and the fact

$$i^n_{max} - i^n_{min} = -\chi(\bar{S}_n) + \tau(n) = -\chi(\bar{S}_+) + nq$$

in Remark 3.4, we have

$$i^n_{max} - Q_n = i^n_{max} - (P_n - q + \tau(+)) = i^n_{max} - i^n_{min} - nq = -\chi(\bar{S}_+) = \rho_n - i^n_{min}. \tag{3.3}$$

Those equations are used in the proof of Lemma 3.15. This motivates the definition of Q_n, which is only for the convenience of the computation. The following remark implies we can freely choose Q_n so that $Q_n - i \in (\rho_n, P_n)$ for $i = 1, \ldots, q$ to carry out the construction.

Remark 3.13. From Lemma 3.5 and the fact

$$P_{n+1} - P_n = i^{n+1}_{min} - i^n_{min} + q = i^{n+1}_{max} - i^n_{max},$$

the isomorphism class of $\mathcal{SHG}_\alpha(-M,-\gamma,i)$ is independent of the choice of the large integer n. Also, by Lemma 3.10, the isomorphism class of $\mathcal{SHG}_\alpha(-M,-\gamma)$ would be the same (up to a $\mathbb{Z}_q$ grading shift) if we consider arbitrary q many consecutive gradings within the range (ρ_n, P_n).

Remark 3.14. For a rationally null-homologous knot $\widehat{K} \subset \widehat{Y}$ with a basepoint p, we can remove a neighborhood of p add a suture δ on $\partial N(p)$ such that two intersection points of $\widehat{K}$ and $\partial N(p)$ lie on $R_+(\gamma)$ and $R_-(\gamma)$, respectively. Then $\widehat{K}$ becomes a vertical tangle α in $(\widehat{Y} - \mathrm{int}N(p), \delta)$ which is rationally null-homologous. In this case, $\mathcal{SHG}_\alpha(\widehat{Y} - \mathrm{int}N(p), \delta, i)$ reduces to $\mathcal{I}_+(-\widehat{Y}, \widehat{K}, i)$ in [43, Definition 4.21], up to a $\mathbb{Z}_q$ grading shift.

Lemma 3.15. Let $\mathcal{SHG}_\alpha(-M,-\gamma)$ be defined as in Definition 3.11. We have

$$\dim_{\mathbb{F}} \mathcal{SHG}_\alpha(-M,-\gamma) = \dim_{\mathbb{F}} \underline{SHG}(-M,-\gamma).$$

Proof. Based on Lemma 3.3, the proof is similar to that of [43, Lemma 4.25]. Now we split the bypass exact triangle of $(-\Gamma_+, -\Gamma_n, -\Gamma_{n+1})$ into five blocks of sizes

$$q, -\chi(\bar{S}_+) + 1, \chi(\bar{S}_+) + (n-1)q - 1, q, -\chi(\bar{S}_+) + 1,$$

respectively, and split the bypass exact triangle of $(-\Gamma_-, -\Gamma_n, -\Gamma_{n+1})$ into five blocks of sizes

$$-\chi(\bar{S}_+) + 1, q, \chi(\bar{S}_+) + (n-1)q - 1, -\chi(\bar{S}_+) + 1, q,$$

respectively. Remark 3.12 ensures that the proof of [43, Lemma 4.25] applies verbatim. Here we only include some key steps as follows.

Suppose $n \in \mathbb{N}$ satisfies $Q_n - \rho_n > q$. We can apply Proposition 3.3. Using blocks, we have the following. (There is no enough room for writing down the whole notation for formal sutured homology, so we will only write down the sutures to denote them.)

size	$-\Gamma_+ \xrightarrow{\psi^+_{+,n}}$	$-\Gamma_n \xrightarrow{\psi^n_{+,n+1}}$	$-\Gamma_{n+1} \xrightarrow{\psi^{n+1}_{+,+}}$	$-\Gamma_+$
q	G		X_1	G
$-\chi(\bar{S}_+) + 1$	H	A	X_2	H
$\chi(\bar{S}_+) + (n-1)q - 1$		E	X_3	
q		F	X_4	
$-\chi(\bar{S}_+) + 1$		D	X_5	

The empty block implies the summands in the block are zeros. Note that

$$i^+_{max} - i^+_{min} + 1 = -\chi(\bar{S}_+) + \tau(+) + 1 \le q + (-\chi(\bar{S}_+) + 1).$$

From the exactness, we know that

$$X_1 = G,\ X_3 = E,\ X_4 = F,\ \text{and } X_5 = D.$$

There is another bypass exact triangle, and similarly we have

$$\begin{array}{ccccccc}
\text{size} & -\Gamma_- & \xrightarrow{\psi^-_{-,n}} & -\Gamma_n & \xrightarrow{\psi^n_{-,n+1}} & -\Gamma_{n+1} & \xrightarrow{\psi^{n+1}_{-,-}} & -\Gamma_- \\
-\chi(\bar{S}_+)+1 & & & A & & A & & \\
q & & & B & & B & & \\
\chi(\bar{S}_+)+(n-1)q-1 & & & C & & C & & \\
-\chi(\bar{S}_+)+1 & I & & D & & X_6 & & I \\
q & J & & & & J & & J
\end{array}$$

Note that

$$i^-_{max} - i^-_{min} + 1 = -\chi(\bar{S}_+) - q + 1 \leq q + (-\chi(\bar{S}_+) + 1).$$

Comparing the two expressions of $\underline{SHG}(-M_T, -\Gamma_{n+1}, S_n)$, we have

$$\begin{pmatrix} G \\ X_2 \\ E \\ F \\ D \end{pmatrix} = \underline{SHG}(-M_T, -\Gamma_{n+1}, S_n) = \begin{pmatrix} A \\ B \\ C \\ X_6 \\ J \end{pmatrix}.$$

Taking sizes into consideration, we know that

$$\begin{pmatrix} G \\ X_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix},\ E = C,\ \text{and} \begin{pmatrix} F \\ D \end{pmatrix} = \begin{pmatrix} X_6 \\ J \end{pmatrix}.$$

Thus, we know that

$$\underline{SHG}(-M_T, -\Gamma_{n+1}, S_n) = \begin{pmatrix} A \\ B \\ E \\ F \\ D \end{pmatrix}.$$

By construction, we have

$$\begin{aligned}\dim_{\mathbb{F}} \mathcal{SHG}_{\alpha}(-M,-\gamma) &= \dim_{\mathbb{F}} B\\ &= \dim_{\mathbb{F}} \underline{SHG}(-M_T,-\Gamma_{n+1}) - \dim_{\mathbb{F}} \underline{SHG}(-M_T,-\Gamma_n)\\ &= \dim_{\mathbb{F}} \underline{SHG}(-M,-\gamma).\end{aligned}$$

Note that the last equality follows from Lemma 3.8. Hence, we obtain the desired result. ■

Remark 3.16. The essential difference for the case of tangles is that Γ_+ is not equal to Γ_-, though it is true in the case of knots in Remark 3.14.

Proposition 3.17. Suppose $n \in \mathbb{N}$ is large enough. Then the map F_n in Lemma 3.6 restricted to $\mathcal{SHG}_{\alpha}(-M,-\gamma)$ is an isomorphism, that is,

$$F_n|_{\mathcal{SHG}_{\alpha}(-M,-\gamma)} : \mathcal{SHG}_{\alpha}(-M,-\gamma) \xrightarrow{\cong} \underline{SHG}(-M,-\gamma).$$

Proof. Based on Lemma 3.8, the proof is similar to that of [43, Proposition 4.26]. It suffices to show that the restriction of F_n is surjective. Here we only include some key steps as follows.

By Lemma 3.8, we know that F_n is surjective. Then it suffices to show that F_n remains surjective when restricted to $\mathcal{SHG}_{\alpha}(-M,-\gamma)$. For any $x \in \underline{SHG}(-M,-\gamma)$, let $y \in \underline{SHG}(-M_T,-\Gamma_n)$ be an element so that $F_n(y) = x$. Suppose

$$y = \sum_{j\in\mathbb{Z}} y_j, \text{ where } y_j \in \underline{SHG}(-M_T,-\Gamma_n,S_n,j).$$

For any y_j, we want to find $y_j' \in \mathcal{SHG}_{\alpha}(-M,-\gamma)$ so that $F_n(y_j) = F_n(y_j')$.

To do this, we first assume that $j \geq Q_n$. Then there exists an integer m so that

$$Q_n - q \leq j - mq \leq Q_n - 1.$$

We can take

$$y_j' = (\psi_{-,n+1}^{n,j-mq})^{-1} \circ \cdots \circ (\psi_{-,n+m}^{n,i_{max}^{n+m}-i_{max}^{n}+j-mq})^{-1} \circ \psi_{+,n+m}^{n+m-1} \circ \cdots \circ \psi_{+,n+1}^{n}(y_j). \tag{3.4}$$

From Lemma 3.5, all the negative bypass maps involved in (3.4) are isomorphisms so the inverses exist. Also, we have

$$y_j' \in \underline{SHG}(-M_T,-\Gamma_n,S_n,j-mq) \subset \mathcal{SHG}_{\alpha}(-M,-\gamma).$$

Finally, from commutative diagrams in Lemma 3.6, we know that $F_n(y_j') = F_n(y_j)$.

For

$$j \in [Q_n - q, Q_n - 1],$$

we can simply take $y_j' = y_j$.

For $j < Q_n - q$, we can pick y_j' similarly, while switching the roles of $\psi_{+,*}^*$ and $\psi_{-,*}^*$ in (3.4).

In summary, we can take

$$y' = \sum_{j\in\mathbb{Z}} y_j' \in \mathcal{SHG}_\alpha(-M,-\gamma) \text{ with } F_n(y') = F_n(y) = x.$$

Hence F_n is surjective, and we obtain the desired result. ∎

Remark 3.18. In Definition 3.11, we use a large enough integer n to define $\mathcal{SHG}_\alpha(-M,-\gamma)$. We can also define Γ_{-n} from Γ_0 by twisting along γ_1 for n times. For a large enough integer n, we can define a vector space $\mathcal{SHG}'_\alpha(-M,-\gamma)$ generalizing $\mathcal{I}_-(-\widehat{Y},\widehat{K})$ in [43, Definition 4.27]. However, from the discussion in [43, Section 4.4 in ArXiv version 2] between $\mathcal{I}_+(-\widehat{Y},\widehat{K})$ and $\mathcal{I}_-(-\widehat{Y},\widehat{K})$, we expect that $\mathcal{SHG}'_\alpha(-M,-\gamma)$ is isomorphic to $\mathcal{SHG}_\alpha(-M,-\gamma)$ up to a $\mathbb{Z}_q$ grading shift. Hence, there is no new information and we skip the discussion here.

3.3 More tangle components

In this subsection, we obtain a decomposition of twisted sutured homology associated to more tangle components. Suppose (M,γ) is a balanced sutured manifold. For a vertical tangle T in M, let $M_T = M\backslash \mathrm{int}N(T)$ and let γ_T be the union of γ and positively oriented meridians of components of T.

First, we prove some lemmas about homology groups.

Lemma 3.19. For any connected tangle α in M, we have

$$\mathrm{rk}_{\mathbb{Z}} H_1(M_\alpha) = \begin{cases} \mathrm{rk}_{\mathbb{Z}} H_1(M) & \text{if } [\alpha] \neq 0 \in H_1(M,\partial M;\mathbb{Q}), \\ \mathrm{rk}_{\mathbb{Z}} H_1(M) + 1 & \text{if } [\alpha] = 0 \in H_1(M,\partial M;\mathbb{Q}). \end{cases}$$

Proof. Consider the long exact sequence associated to the pair (M, M_α):

$$H^1(M,M_\alpha) \xrightarrow{p_1^*} H^1(M) \xrightarrow{i_1^*} H^1(M_\alpha) \xrightarrow{\delta_1^*} H^2(M,M_\alpha) \xrightarrow{p_2^*} H^2(M) \xrightarrow{i_2^*} H^2(M_\alpha) \xrightarrow{\delta_2^*} H^3(M,M_\alpha). \tag{3.5}$$

By the excision theorem, we have

$$H^*(M, M_\alpha) \cong H^j(N(\alpha), \partial N(\alpha) \cap M_\alpha) \cong H^j(D^2, \partial D^2) \cong \begin{cases} \mathbb{Z} & j = 2, \\ 0 & j = 1, 3. \end{cases}$$

Since $H^2(N(\alpha), \partial N(\alpha) \cap M_\alpha)$ is generated by the disk that is the Poincaré dual of $[\alpha \cap N(\alpha)]$ and p_2^* is induced by the projection, the image of p_2^* is generated by the Poincaré dual of $[\alpha]$. Since $H^1(M)$ and $H_1(M)$ always have the same rank, we obtain the rank equation from (3.5). ■

Lemma 3.20. Suppose (M, γ) is a balanced sutured manifold. There exists a (possibly empty) tangle $T = T_1 \cup \cdots \cup T_m$ in M, such that $\mathrm{Tor}(H_1(M_T)) = 0$ and for any $T' \subset T$ and $T_i \subset T \backslash T'$, we have

$$[T_i] = 0 \in H_1(M_{T'}, \partial M_{T'}; \mathbb{Q}). \tag{3.6}$$

Proof. Suppose α is a connected tangle in M. From (3.5) and the proof of Lemma 3.19, we have

$$\mathbb{Z}\langle \phi_\alpha \rangle \xrightarrow{p_2^*} H^2(M) \xrightarrow{i_2^*} H^2(M_\alpha) \to 0,$$

where ϕ_α is the Poincaré dual of $[\alpha]$. By the universal coefficient theorem, the torsion subgroups of $H^2(M)$ and $H_1(M)$ are isomorphic. In particular, $\mathrm{Tor}(H^2(M)) = 0$ if and only if $\mathrm{Tor}(H_1(M)) = 0$. Let α be a rationally null-homologous tangle, then

$$\mathrm{Tor}(H^2(M_\alpha)) \cong \mathrm{Tor}(H^2(M))/\mathrm{PD}(\alpha).$$

Thus, we can always choose connected tangles

$$T_1 \subset M, T_2 \subset M_{T_1}, T_3 \subset M_{T_1 \cup T_2}, \ldots, T_m \subset M_{T_1 \cup \cdots \cup T_{m-1}}$$

that are rationally null-homologous to kill the whole torsion subgroup. In other word, for $T = T_1 \cup \cdots \cup T_m$, we have $\mathrm{Tor}(H_1(M_T)) = 0$.

By Lemma 3.19, we have

$$\mathrm{rk}_{\mathbb{Z}} H_1(M_T) = \mathrm{rk}_{\mathbb{Z}} H_1(M) + m. \tag{3.7}$$

Hence for any T' and any T_i satisfy the assumption, (3.6) holds, otherwise it contradicts with the rank equality (3.7). ■

Remark 3.21. Since moving the endpoints of a tangle on the boundary of the ambient 3-manifold does not change the homology class of the tangle, we can suppose the tangle

T in Lemma 3.20 is a vertical tangle. Moreover, when M has connected boundary, we can suppose endpoints of T all lie in a neighborhood of a point on the suture γ.

Lemma 3.22. Suppose (M,γ) is a balanced sutured manifold and suppose α is a connected rationally null-homologous tangle of order q. Let S_α be a Seifert surface of T_i, that is, ∂S_i consists of q parallel copies of α and arcs on ∂M. Suppose $S_1,\ldots,S_n$ are admissible surfaces in (M,γ) generating $H_2(M,\partial M)$. Then the restrictions of $S_1,\ldots,S_n$ and S_α on M_T generate $H_2(M_T,\partial M_T)$.

Proof. From (3.5) and the proof of Lemma 3.19, we have

$$0\to H^1(M)\xrightarrow{i_1^*}H^1(M_\alpha)\xrightarrow{\delta_1^*}\mathbb{Z}\langle\phi_\alpha\rangle\xrightarrow{p_2^*}H^2(M),$$

where ϕ_α is the Poincaré dual of $[\alpha]$. It is straightforward to calculate

$$\delta_1^*(\mathrm{PD}([S_\alpha]))=q\phi_\alpha. \tag{3.8}$$

Since $H^1(M)\cong H_2(M,\partial M)$, we have

$$\begin{aligned}H_2(M_\alpha,\partial M_\alpha)/H_2(M,\partial M)\cong&H^1(M_\alpha)/H^1(M)\cong H^1(M_\alpha)/\mathrm{im}(i_1^*)\\ \cong H^1(M_\alpha)/\ker(\delta_1^*)\cong&\mathrm{im}(\delta_1^*)\cong\ker(p_2^*).\end{aligned} \tag{3.9}$$

Since the image of p_2^* is the Poincaré dual of $[\alpha]$, we have

$$\ker(p_2^*)\cong\langle q\phi_\alpha\rangle. \tag{3.10}$$

Combining (3.8), (3.9), and (3.10), we know that $[S_\alpha]$ generates $H_2(M_\alpha,\partial M_\alpha)/H_2(M,\partial M)$. Thus, we conclude the desired property. ■

In the rest of this subsection, we suppose (M,γ) is a balanced sutured manifold and $T=T_1\cup\cdots\cup T_m$ is a vertical tangle satisfying Lemma 3.20. Suppose the order of the first component T_1 in $H_1(M)$ is q_1 and suppose S_1 is a Seifert surface of T_1.

Convention. We will still use S_1 to denote its restriction on M_{T_1}. This also applies to other Seifert surfaces mentioned below.

We adapt the construction in Subsection 3.1. Applying results in Subsection 3.2, we have

$$\mathcal{SHG}_{T_1}(-M,-\gamma):=\bigoplus_{i=1}^{q_1}\underline{SHG}(-M_{T_1},-\Gamma_n,(S_1)_n,O_n-i)\cong\underline{SHG}(-M,-\gamma),$$

where n is a large integer, $(S_1)_n$ is a (possibly empty) stabilization of S_1, and Q_n is a fixed integer. For simplicity, we choose a large integer n_1 such that $(S_1)_{n_1} = S_1$ and write

$$\Gamma^1_{n_1} = \Gamma_n|_{n=n_1} \text{ and } Q^1_{n_1} = Q_n|_{n=n_1}.$$

For the second component T_2, suppose S_2 is its Seifert surface in M_{T_1} with ∂S^2 containing q_2 copies of T_2. Now we can apply the construction in Subsection 3.1 and the results in Subsection 3.2 to $(M, \Gamma^1_{n_1})$. For a large integer n_2 such that $(S_2)_{n_1} = S_2$, we define

$$\begin{aligned}\mathcal{SHG}_{T_1\cup T_2}(-M,-\gamma) &:= \bigoplus_{i_1=1}^{q_1}\bigoplus_{i_2=1}^{q_2} \underline{SHG}(-M_{T_1\cup T_2}, -\Gamma^2_{n_2}, (S_1,S_2), (Q^1_{n_1}-i_1, Q^2_{n_2}-i_2)) \\ &\cong \underline{SHG}(-M,-\gamma).\end{aligned}$$

Iterating this procedure, we have the following definition.

Definition 3.23. For $i = 1,\dots,m$, suppose the component T_k is rationally null-homologous of order q_k in $M_{T_1\cup\cdots\cup T_{k-1}}$. Inductively, for $k = 1,\dots,m$, we choose a large integer n_k, a suture $\Gamma^k_{n_k} \subset \partial M_{T_1\cup\cdots\cup T_k}$, a Seifert surface $S_k = (S_k)_{n_k} \subset M_{T_1\cup\cdots\cup T_k}$, and an integers $Q^k_{n_k}$, such that $n_k, \Gamma^k_{n_k}, S_k, Q^k_{n_k}$ depend on the choices for the first $(k-1)$ tangles. Define

$$\mathcal{SHG}_T(-M,-\gamma) := \bigoplus_{i_1\in[1,q_1],\dots,i_m\in[1,q_m]} \underline{SHG}(-M_T, -\Gamma^m_{n_m}, (S_1,\dots,S_m), (Q^1_{n_1}-i_1,\cdots,Q^m_{n_m}-i_m)).$$

Remark 3.24. Though we only use the subscript T in the notation $\mathcal{SHG}_T(-M,-\gamma)$, it is not known if $\mathcal{SHG}_T(-M,-\gamma)$ is independent of the choices of all constructions. In particular, we have to choose an order of the components to define $\mathcal{SHG}_T(-M,-\gamma)$.

Applying results in Subsection 3.2 for m times, the following proposition is straightforward.

Proposition 3.25. $\mathcal{SHG}_T(-M,-\gamma) \cong \underline{SHG}(-M,-\gamma)$.

The map $H_1(M_{T_1}) \to H_1(M)$ is surjective. The q_1 direct summands of $\underline{SHG}_{T_1}(-M,-\gamma)$ correspond to the order q_1 torsion subgroup generated by

$$[T_1] \in \mathrm{Tor}(H_1(M,\partial M)) \cong \mathrm{Tor}(H^2(M)) \cong \mathrm{Tor}(H_2(M)).$$

Hence, the summands of $\underline{SHG}_{T_1}(-M,-\gamma)$ provide a decomposition of $\underline{SHG}(-M,-\gamma)$ with respect to the torsion subgroup generated by $[T_1]$. By induction and the fact that

$\mathrm{Tor}(H_1(M_T)) = 0$, we can regard summands in $\mathcal{SHG}_T(-M,-\gamma)$ as a decomposition of $\underline{SHG}(-M,-\gamma)$ with respect to $\mathrm{Tor}(H_1(M))$.

To provide a decomposition of $\underline{SHG}(-M,-\gamma)$ with respect to the whole $H_1(M)$ as in Theorem 1.1, we choose admissible surfaces $S_{m+1},\ldots,S_{m+n}$ generating $H_2(M,\partial M)$. By Lemma 3.22, the restrictions of $S_1,\ldots,S_{m+n}$ generate $H_2(M_T,\partial M_T)$. By Lemma 3.9, the gradings associated to these surfaces behave well under restriction.

Definition 3.26. Consider the construction as above. For $i = 1,\ldots,m+n$, let $\rho_1,\ldots,\rho_{m+n} \in H_1(M_T) = H_1(M_T)/\mathrm{Tors}$ be the class satisfying $\rho_i \cdot S_j = \delta_{i,j}$. Consider

$$j_* : \mathbb{Z}[H_1(M_T)] \to \mathbb{Z}[H_1(M)].$$

We write

$$H = H_1(M), \boldsymbol{S} = (S_1,\ldots,S_{m+n}), -i_k' = O_{n_k}^k - i_{n+k} \text{ for } k = 1,\ldots,m,$$

and

$$-\boldsymbol{i}' = (-i_1',\ldots,-i_m',-i_{m+1},\ldots,-i_{m+n}), \boldsymbol{\rho}^{-\boldsymbol{i}'} = \rho_1^{-i_1'}\cdots\rho_n^{-i_m'}\cdot\rho_{m+1}^{-i_{n+1}}\cdots\rho_{m+n}^{-i_{m+n}}.$$

The **enhanced Euler characteristic** of $\underline{SHG}(-M,-\gamma)$ is

$$\begin{aligned}\chi_{\mathrm{en}}(\underline{SHG}(-M,-\gamma)) &= j_*(\chi(\mathcal{SHG}_T(-M,-\gamma)))\\ &:= j_*\Big(\sum_{\substack{i_1\in[1,q_1],\ldots,i_m\in[1,q_m]\\ (i_{m+1},\ldots,i_{m+n})\in\mathbb{Z}^n}} \chi(\underline{SHG}(-M_T,-\gamma_T,\boldsymbol{S},-\boldsymbol{i}'))\cdot\boldsymbol{\rho}^{-\boldsymbol{i}'}\Big) \in \mathbb{Z}[H]/\pm H.\end{aligned}$$

For $h \in H_1(M)$, let $\underline{SHG}(-M,-\gamma,h)$ be image of the summand of $\mathcal{SHG}_T(-M,-\gamma)$ under the isomorphism in Proposition 3.25 whose corresponding element in $\chi_{\mathrm{en}}(\underline{SHG}(-M,-\gamma))$ is h.

Remark 3.27. As mentioned in Remark 3.24, the definition of $\underline{SHG}(-M,-\gamma,h)$ is not canonical, that is, it may depend on many auxiliary choices. After fixing these choices, it is still only well-defined up to a global grading shift by multiplication by an element in $h_0 \in H_1(M)$. However, by Theorem 5.1, the enhanced Euler characteristic $\chi_{\mathrm{en}}(\underline{SHG}(-M,-\gamma))$ only depends on (M,γ).

4 Sutured Heegaard Floer Homology

In this section, we discuss properties of sutured (Heegaard) Floer homology *SFH* that are similar to those for twisted sutured homology, so we can apply results in Section 3 to *SFH*. Since *SFH* is not defined by closures of sutured manifolds, the maps associated to surgeries and contact handle attachments are different from those for $\underline{SHG}$.

4.1 Construction and gradings

In this subsection, we describe the definition of *SFH* and discuss the gradings on *SFH* associated to admissible surfaces.

Definition 4.1 ([29, Section 2]). A **balanced diagram** $\mathcal{H} = (\Sigma, \alpha, \beta)$ is a tuple satisfying the following.

(1) Σ is a compact, oriented surface with boundary.
(2) $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ and $\beta = \{\beta_1, \ldots, \beta_n\}$ are two sets of pairwise disjoint simple closed curves in the interior of Σ.
(3) The maps $\pi_0(\partial\Sigma) \to \pi_0(\Sigma\backslash\alpha)$ and $\pi_0(\partial\Sigma) \to \pi_0(\Sigma\backslash\beta)$ are surjective.

For such triple, let N be the 3-manifold obtained from $\Sigma \times [-1, 1]$ by attaching 3-dimensional 2-handles along $\alpha_i \times \{-1\}$ and $\beta_i \times \{1\}$ for $i = 1, \ldots, n$ and let $\nu = \partial\Sigma \times \{0\}$. A balanced diagram (Σ, α, β) is called **compatible** with a balanced sutured manifold (M, γ) if the balanced sutured manifold (N, ν) is diffeomorphic to (M, γ).

Suppose $\mathcal{H} = (\Sigma, \alpha, \beta)$ is a balanced diagram with $g = g(\Sigma)$ and $n = |\alpha| = |\beta|$.

Convention. In this paper, we always suppose balanced diagrams satisfy the admissible condition in [29, Section 3].

Consider two tori

$$\mathbb{T}_\alpha := \alpha_1 \times \cdots \times \alpha_n \text{ and } \mathbb{T}_\beta := \beta_1 \times \cdots \times \beta_n$$

in the symmetric product

$$\mathrm{Sym}^n\Sigma := (\prod_{i=1}^n \Sigma)/S_n.$$

The chain complex $SFC(\mathcal{H})$ is a free $\mathbb{F}_2$-module generated by intersection points $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Let $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ be the set of homology classes of Whitney disks connecting intersection points $\boldsymbol{x}$ and $\boldsymbol{y}$. Choose a generic path of almost complex structures J_s on $\mathrm{Sym}^n\Sigma$. For $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, let $\mathcal{M}_{J_s}(\phi)$ be the moduli space of J_s-holomorphic maps

$$u : [0, 1] \times \mathbb{R} \to \mathrm{Sym}^n\Sigma,$$

which represent ϕ and let $\mu(\phi)$ be the expected dimension of $\mathcal{M}_{J_s}(\phi)$. The moduli space $\mathcal{M}_{J_s}(\phi)$ has a natural action of $\mathbb{R}$, corresponding to reparametrization of the source. We write

$$\widehat{\mathcal{M}}_{J_s}(\phi) := \mathcal{M}_{J_s}(\psi)/\mathbb{R}.$$

Based on the above construction, Juhász [29] defined a differential on $SFC(\mathcal{H})$ by

$$\partial_{J_s}(\boldsymbol{x}) = \sum_{\boldsymbol{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\substack{\phi\in\pi_2(\boldsymbol{x},\boldsymbol{y})\\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}_{J_s}(\phi)\cdot \boldsymbol{y}.$$

Theorem 4.2 ([29, 33]). Suppose (M,γ) is a balanced sutured manifold. Then there is an admissible balanced diagram $\mathcal{H}$ compatible with (M,γ). The vector spaces $H(SFC(\mathcal{H}),\partial_{J_s})$ for different choices of $\mathcal{H}$ and J_s, together with some canonical maps, form a transitive system $SFH(M,\gamma)$ over $\mathbb{F}_2$.

For a balanced sutured manifold (M,γ), we can decompose $SFH(M,\gamma)$ along spin^c structures.

Fix a Riemannian metric g on M. Let v_0 be a nowhere vanishing vector field along ∂M that points into M along $R_-(\gamma)$, points out of M along $R_+(\gamma)$, and on γ it is the gradient of the height function $A(\gamma)\times I\to I$. The space of such vector fields is contractible, so the choice of v_0 is not important.

Suppose v and w are nowhere vanishing vector fields on M that agree with v_0 on ∂M. They are called **homologous** if there is an open ball $B\subset \mathrm{int}M$ such that v and w are homotopic on $M\backslash B$ through nowhere vanishing vector fields rel ∂M. Let $\mathrm{Spin}^c(M,\gamma)$ be the set of homology classes of nowhere vanishing vector fields v on M with $v|_{\partial M}=v_0$. Note that $\mathrm{Spin}^c(M,\gamma)$ is an affine space over $H^2(M,\partial M)$.

Suppose $\mathcal{H}=(\Sigma,\alpha,\beta)$ is a balanced diagram compatible with (M,γ). For each intersection point $\boldsymbol{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, we can assign a spin^c structure $\mathfrak{s}(\boldsymbol{x})\in \mathrm{Spin}^c(M,\gamma)$ as follows (cf. [29, Section 4]).

We choose a self-indexing Morse function $f: M\to[-1,4]$ such that

$$f^{-1}(\frac{3}{2})=\Sigma\times\{0\}.$$

Moreover, curves α,β are intersections of $\Sigma\times\{0\}$ with the ascending and descending manifolds of the index 1 and 2 critical points of f, respectively. Then any intersection point of $\alpha_i\subset\alpha$ and $\beta_j\subset\beta$ corresponds to a trajectory of grad$f$ connecting a index 1 critical point to a index 2 critical point. For $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, let $\gamma_{\boldsymbol{x}}$ be the multi-trajectory corresponding to intersection points in $\boldsymbol{x}$.

In a neighborhood $N(\gamma_{\boldsymbol{x}})$, we can modify grad$f$ to obtain a nowhere vanishing vector field v on M such that $v|_{\partial M} = v_0$. Let $\mathfrak{s}(\boldsymbol{x}) \in \mathrm{Spin}^c(M,\gamma)$ be the homology class of this vector field v.

From the assignment of the spin^c structure, we have the following proposition.

Proposition 4.3. For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we have

$$\mathfrak{s}(\boldsymbol{x}) - \mathfrak{s}(\boldsymbol{y}) = \mathrm{PD}([\gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{y}}]),$$

where $\mathrm{PD} : H_1(M) \to H^2(M, \partial M)$ is the Poincaré duality map.

It can be shown that there is no differential between generators corresponding to different spin^c structures. Hence, we have the following decomposition.

Proposition 4.4 ([29]). For any balanced sutured manifold (M,γ), there is a decomposition

$$SFH(M,\gamma) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(M,\partial M)} SFH(M,\gamma,\mathfrak{s}).$$

Suppose $S \subset (M,\gamma)$ is an admissible surface S. To associate a $\mathbb{Z}$-grading on $SFH(M,\gamma)$ similar to Subsection 2.1, we need to suppose (M,γ) is **strongly balanced**, that is, for every component F of ∂M, we have

$$\chi(F \cap R_+(\gamma)) = \chi(F \cap R_-(\gamma)).$$

Remark 4.5. If ∂M is connected, then it is automatically strongly balanced. For any balanced sutured manifold (M,γ), we can obtain a strongly balanced manifold (M',γ') by attaching contact 1-handles [30, Remark 3.6]. In Subsection 4.4, we will show

$$SFH(M',\gamma') \cong SFH(M,\gamma)$$

and this isomorphism respects spin^c structures. Hence, we can always deal with a strongly balanced manifold without losing any information.

Convention. When discussing the $\mathbb{Z}$-grading on $SFH(M,\gamma)$ associated to an admissible surface $S \subset (M,\gamma)$, we always suppose (M,γ) is strongly balanced.

The following construction is based on [30, Section 3].

Let $v_0^\perp$ be the plane bundle perpendicular to v_0 under the fixing Riemannian metric g. Suppose $v_0^\perp$ is oriented so that v_0 is the positive normal vector field. The strongly balanced condition on (M,γ) ensures that $v_0^\perp$ is trivial (cf. [30, Proposition 3.4]). Let t be a trivialization of $v_0^\perp$. Since any spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ can be represented by a nonvanishing vector field v on M with $v|_{\partial M} = v_0$, we can define

$$c_1(\mathfrak{s},t) := c_1(v^\perp,t) \in H^2(M,\partial M)$$

to be the relative Euler class of the plane bundle $v^\perp$ with respect to the trivialization t, where $v^\perp$ is perpendicular to v. In other words, the class $c_1(\mathfrak{s},t)$ is the obstruction to extending t from ∂M to a trivialization of $v^\perp$ over M.

Let v_S be the positive unit normal field of S. For a generic S, we can suppose v_S is nowhere parallel to v_0 along ∂S. Let $p(v_S)$ be the projection of v_S into $v_0^\perp$. Note that $p(v_S)|_{\partial S}$ is nowhere zero. Suppose the components of ∂S are $T_1,\dots,T_k$, oriented by the boundary orientation.

For $i = 1,\dots,k$, let $r(T_i,t)$ be the rotation number $p(v_S)|_{T_i}$ with respect to the trivialization t as we go around T_i. Moreover, define

$$r(S,t) := \sum_{i=1}^{k} r(T_i,t).$$

Suppose $T_1,\dots,T_k$ intersect γ transversely. Define

$$c(S,t) = \chi(S) - \frac{1}{2}|\partial S \cap \gamma| - r(S,t). \tag{4.1}$$

Remark 4.6. The original definition of $c(S,t)$ in [30, Section 3] involves the index $I(S)$, which is equal to $\frac{1}{2}|\partial S \cap \gamma|$ when $T_1,\dots,T_k$ intersect γ transversely (cf. [30, Lemma 3.9]).

Suppose t_S is the trivialization of $v_0^\perp$ induced by $p(v_S)|_{\partial S}$. Then for any $v^\perp$ with $v^\perp|_{\partial M} = v_0^\perp$ and any trivialization t of $v_0^\perp$, we have

$$\langle c_1(v^\perp,t_S) - c_1(v^\perp,t), [S]\rangle = r(S,t) \tag{4.2}$$

(cf. Proof of [30, Lemma 3.10]; see also [31, Lemma 3.11]). In particular, we have $r(S,t_S) = 0$.

Definition 4.7. Consider the construction as above. Define

$$SFH(M,\gamma,S,i) := \bigoplus_{\substack{\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma)\\ \langle c_1(\mathfrak{s},t_S),[S]\rangle = -2i}} SFH(M,\gamma,\mathfrak{s}). \tag{4.3}$$

Remark 4.8. The minus sign of $(2i)$ is to make this definition parallel to the $\mathbb{Z}$-grading on $\underline{SHG}(M,\gamma)$ associated to S. See the proofs of the following propositions.

Proposition 4.9. The decomposition in Definition 4.7 satisfies properties in Theorem 2.4, replacing $\underline{SHG}$ by SFH.

Proof. Term (1) follows from the adjunction inequality in [31, Theorem 2]. Note that if $2i = |\partial S \cap \gamma| - \chi(S)$, then for $\mathfrak{s}$ corresponds to $SFH(M,\gamma,S,i)$, we have

$$\langle c_1(\mathfrak{s},t_S),[S]\rangle = \chi(S) - |\partial S \cap \gamma| = c(S,t_S), \tag{4.4}$$

where the last equality follows from (4.1) and (4.2).

Term (2) follows from [30, Lemma 3.10] and (4.4).

Terms (3)–(5) follow from definitions and symmetry on balanced diagrams. ■

Proposition 4.10. Consider the stabilized surfaces S^p and S^{p+2k} in Theorem 2.8. Then for any $l \in \mathbb{Z}$, we have

$$SFH(M,\gamma,S^p,l) = SFH(M,\gamma,S^{p+2k},l+k).$$

Proof. Suppose S^+ and S^- are positive and negative stabilizations of S. Since the stabilization operation is local, we have the following equation by direct calculation

$$r(S^+,t) = r(S,t) - 1$$

for any trivialization t of $v_0^{\perp}$; see Figure 9.

Note that $[S^+] = [S]$. Hence for $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ corresponds to $SFH(M,\gamma,S,i)$, we have

$$\begin{aligned}
\langle c_1(\mathfrak{s},t_{S^+}),[S^+]\rangle &= \langle c_1(\mathfrak{s},t_S),[S^+]\rangle + r(S^+,t_S)\\
&= \langle c_1(\mathfrak{s},t_S),[S^+]\rangle + r(S,t_S) - 1\\
&= \langle c_1(\mathfrak{s},t_S),[S^+]\rangle - 1\\
&= \langle c_1(\mathfrak{s},t_S),[S]\rangle - 1\\
&= -2i - 1.
\end{aligned}$$

Applying this calculation for $(2k)$ times gives the desired result. ■

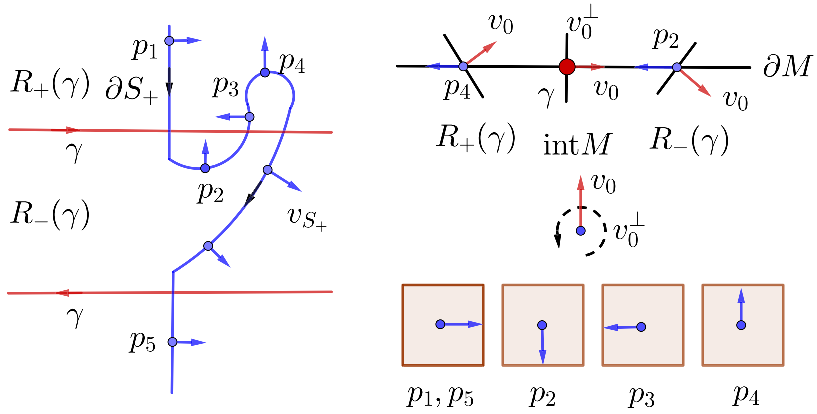

Fig. 9. Left subfigure, the positive stabilization S_+ and the vector v_{S_+}, where int$M$ is inside the page. Right top subfigure, the vector field v_0 and the plane field $v_0^\perp$ from another viewpoint. Right middle subfigure, the orientation of $v_0^\perp$, where v_0 points out the page. Right bottom subfigures, the projections of v_{S_+} on $v_0^\perp$ on given points.

Proposition 4.11. Suppose S_1 and S_2 are two admissible surfaces in (M,γ) such that

$$[S_1] = [S_2] = \alpha \in H_2(M, \partial M).$$

Then there exists a constant C so that

$$\underline{SHG}(M,\gamma,S_1,l) = \underline{SHG}(M,\gamma,S_2,l+C).$$

Proof. This follows directly from the definition. ■

4.2 Euler characteristics

Then we can consider the Euler characteristic of *SFH* with respect to spin^c structures.

Definition 4.12 (). For a balanced sutured manifold (M,γ), let the $\mathbb{Z}_2$ grading of $SFH(M,\gamma)$ be induced by the sign of intersection points of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ for some compatible diagram $\mathcal{H} = (\Sigma,\alpha,\beta)$ (cf. [18, Section 3.4]). Suppose $H = H_1(M)$ and choose any $\mathfrak{s}_0 \in Spin^c(M,\gamma)$. The **Euler characteristic** of $SFH(M,\gamma)$ is

$$\chi(SFH(M,\gamma)) := \sum_{\substack{\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma)\\ \mathfrak{s}-\mathfrak{s}_0=h\in H^2(M,\partial M)}} \chi(SFH(M,\gamma,\mathfrak{s}))\cdot \mathrm{PD}(h) \in \mathbb{Z}[H]/\pm H,$$

where $\mathrm{PD}: H^2(M,\partial M) \to H_1(M)$ is the Poincaré duality map.

Theorem 4.13 ([18]). Suppose (M,γ) is a balanced sutured manifold. Then

$$\chi(SFH(M,\gamma)) = \tau(M,\gamma),$$

where $\tau(M,\gamma)$ is a (Turaev-type) torsion element computed from the map

$$\pi_1(R_-(\gamma)) \to \pi_1(M)$$

by Fox calculus. In particular, if $(M,\gamma) = (Y(K),\gamma_K)$ for a knot K in Y, then

$$\tau(M,\gamma) = (1-[m])\tau(Y(K)),$$

where m is the meridian of K and $\tau(Y(K))$ is the Turaev torsion defined in [60].

4.3 Surgery exact triangle

Suppose (M,γ) is a balanced sutured manifold and K is a knot in M. Consider three balanced sutured manifolds (M_i,γ_i) for $i=1,2,3$ obtained from (M,γ) by Dehn surgeries along K. If the Dehn filling curves $\eta_1,\eta_2,\eta_3 \subset \partial(M\backslash\mathrm{int}\partial N(K))$ satisfy

$$\eta_1\cdot\eta_2 = \eta_2\cdot\eta_3 = \eta_3\cdot\eta_1 = -1,$$

then we have the following exact triangle for twisted sutured homology from the surgery exact triangle in the closure of (M_i,γ_i)

$$\begin{tikzcd} \underline{SHG}(M_1,\gamma_1) \arrow[rr] & & \underline{SHG}(M_2,\gamma_2) \arrow[dl] \\ & \underline{SHG}(M_3,\gamma_3) \arrow[ul] & \end{tikzcd} \tag{4.5}$$

In this subsection, we show the exact triangle (4.5) is also true when replacing $\underline{SHG}$ by SFH.

First, we quickly review Juhász's construction of the cobordism map associated to a Dehn surgery (cf. [32, Section 6], see also [52] for Dehn surgeries on closed 3-manifolds).

For simplicity, suppose η_1 is the meridian of K. Choose an arc a connecting K to $R_+(\gamma)$. We can construct a sutured triple diagram $(\Sigma,\alpha,\beta,\delta)$ satisfying the following properties.

(1) $|\alpha| = |\beta| = |\gamma| = d$.
(2) $(\Sigma, \alpha, \{\beta_2, \dots, \beta_d\})$ is a diagram of $(M', \gamma') = (M\backslash N(K \cup a), \gamma)$.
(3) $\delta_2, \dots, \delta_d$ are obtained from $\beta_2, \dots, \beta_d$ by small isotopy, respectively.
(4) After compressing Σ along $\beta_2, \dots, \beta_d$, the induced curves β_1 and δ_1 lie in the punctured torus $\partial N(K)\backslash N(a)$.
(5) β_1 represents the meridian η_1 of K and δ_1 represents the curve η_2. In particular, β_1 intersects δ_1 transversely at one point.

Then we can construct a 4-manifold $\mathcal{W}_{\alpha,\beta,\delta}$ associated to $(\Sigma, \alpha, \beta, \delta)$ such that it is a cobordism from $(M, \gamma) = (M_1, \gamma_1)$ to

$$(M_2, \gamma_2) \sqcup (R_+ \times I \times \partial R_+ \times I)\#^{d-n}(S^2 \times S^1),$$

where $R_+ = R_+(\gamma)$ and different copies of $S^2 \times S^1$ might be summed along different components of $R_+ \times I$.

Choose a top dimensional generator $\Theta_{\beta,\delta}$ of

$$SFH(R_+ \times I \times \partial R_+ \times I)\#^{d-n}(S^2 \times S^1) \cong \Lambda^* H^1(\#^{d-n}(S^2 \times S^1)).$$

Note that (Σ, α, β) is a balanced diagram of (M_1, γ_1) and (Σ, α, δ) is a balanced diagram of (M_2, γ_2). There is a map

$$F_{\alpha,\beta,\gamma} : SFH(\Sigma, \alpha, \beta) \otimes SFH(\Sigma, \beta, \delta) \to SFH(\Sigma, \alpha, \delta)$$

obtained by counting holomorphic triangles in $(\Sigma, \alpha, \beta, \delta)$. Then define the cobordism map as

$$F_1 : SFH(M_1, \gamma_1) \to SFH(M_2, \gamma_2)$$
$$F_1(x) = F_{\alpha,\beta,\delta}(x, \Theta_{\beta,\delta}).$$

Similarly, we can define the cobordism maps F_2 and F_3.

Theorem 4.14 (Surgery exact triangle). Consider (M_i, γ_i) and cobordism maps F_i for $i = 1, 2, 3$ as above. Then we have an exact triangle

$$\begin{tikzcd} SFH(M_1, \gamma_1) \arrow[rr, "F_1"] & & SFH(M_2, \gamma_2) \arrow[dl, "F_2"] \\ & SFH(M_3, \gamma_3) \arrow[ul, "F_3"] & \end{tikzcd} \tag{4.6}$$

Proof. The proof follows the proof of [49, Theorem 9.12] without essential changes (see also [51, 53]). Since the cobordism maps F_i are well-defined on *SFH*, we can verify the exact triangle for any diagram. We can construct a diagram $(\Sigma, \alpha, \beta, \delta, \zeta)$ such that $(\Sigma, \alpha, \beta, \delta)$ defines F_1, $(\Sigma, \alpha, \delta, \zeta)$ defines F_2, and $(\Sigma, \alpha, \zeta, \beta)$ defines F_3. Then we can verify the assumptions of the triangle detection lemma [51, Lemma 4.2] by counting holomorphic squares and pentagons and then this lemma induces the desired exact triangle. ■

4.4 Contact handles and bypasses

Suppose $(M, \gamma) \subset (M', \gamma')$ is a proper inclusion of balanced sutured manifolds and suppose ξ is a contact structure on $M' \backslash \mathrm{int} M$ with dividing sets $\gamma' \cup (-\gamma)$. Honda, Kazez, and Matić [28] defined a map

$$\Phi_\xi : SFH(M, \gamma) \to SFH(M', \gamma'),$$

which is indeed the motivation of Baldwin and Sivek's construction in Subsection 2.2.

Originally, this map is defined by partial open book decompositions, and there are some technical conditions. Juhász and Zemke [34] provided an alternative description of this map by contact handle decompositions. Their description is explicit on balanced diagrams of sutured manifolds. We will follow this alternative definition and describe the maps for contact 1- and 2-handle attachments.

It is also worth mentioning that Zarev [67] defined a gluing operation for sutured manifolds and conjectured the map associated to contact structures above can be recovered by the gluing operation. This was proved by Leigon and Salmoiraghi [38].

Juhász and Zemke's construction can be shown in Figure 10 and Figure 11 ([34, Figure 1.1]). Note that for all maps associated to contact structures, we should reverse the orientations of the manifold and the suture.

Let (Σ, α, β) be a balanced diagram compatible with (M, γ). Then $(-\Sigma, \alpha, \beta)$ is a balanced diagram compatible with $(-M, -\gamma)$. Attaching a (3-dimensional) contact 1-handle along D_+ and D_- corresponds to attaching a 2-dimensional 1-handle along $D_+ \cap \gamma$ and $D_- \cap \gamma$ in $\partial\Sigma$. This operation does not change the sutured Floer chain complex and we define $C_{h^1} = C_{h^1, D_+, D_-}$ as the tautological map on intersection points.

For a contact 2-handle attachment along $\mu \subset \partial M$, note that $|\mu \cap \gamma| = 2$. Suppose λ_+ and λ_- are arcs corresponding to $\mu \cap R_+(\gamma)$ and $\mu \cap R_-(\gamma)$, respectively. After isotopy, we can suppose λ_+ and λ_- are properly embedded arcs on Σ. We glue a 2-dimensional 1-handle h along $\partial\Sigma$ to obtain Σ', and construct two curves α_0 and β_0 that intersect at

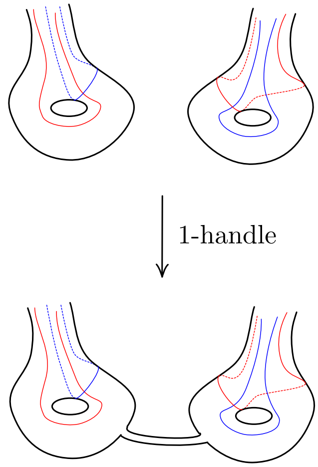

Fig. 10. Contact 1-handle.

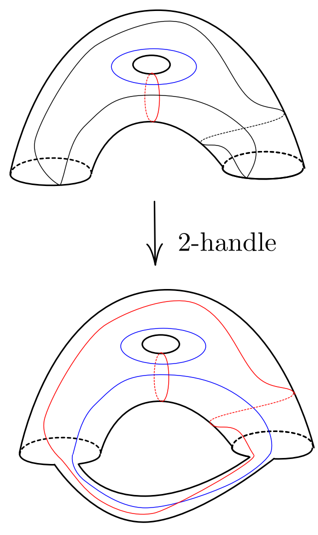

Fig. 11. Contact 2-handle.

one point c in H, and such that

$$\alpha_0 \cap \Sigma = \lambda_+, \beta_0 \cap \Sigma = \lambda_-.$$

Consider the balanced diagram $(\Sigma', \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\})$ and define the map associated to the contact 2-handle attachment as

$$C_{h^2}(\boldsymbol{x}) = C_{h^2,\mu}(\boldsymbol{x}) := \boldsymbol{x} \times \{c\}$$

for any $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Since a bypass attachment can be regarded as a composition of a contact 1-handle and 2-handle attachment (cf. Subsection 2.2), we can define the bypass map by $C_{h^2} \circ C_{h^1}$.

Honda [25] proposed an exact triangle associated to bypass maps for *SFH*, which is indeed the motivation of the bypass exact triangle in Theorem 2.12. A proof of the exact triangle based on bordered sutured Floer homology was provided by Etnyre, Vela-Vick, and Zarev [17].

Theorem 4.15 (Bypass exact triangle, [17, Section 6]). Suppose (M,γ_1), (M,γ_2), (M,γ_3) are balanced sutured manifolds such that the underlying 3-manifolds are the same, and the sutures γ_1, γ_2, and γ_3 only differ in a disk shown in Figure 5. Then there exists an exact triangle

$$\begin{tikzcd} SFH(-M,-\gamma_1) \arrow[rr, "\psi_1"] & & SFH(-M,-\gamma_2) \arrow[dl, "\psi_2"] \\ & SFH(-M,-\gamma_3) \arrow[ul, "\psi_3"] & \end{tikzcd}$$

where ψ_1, ψ_2, ψ_3 are bypass maps associated to the corresponding bypass arcs.

From Juhász and Zemke's description of contact gluing maps, it is obvious that the maps respect the decomposition of *SFH* by spin^c structures. We describe this fact explicitly as follows.

Lemma 4.16. Suppose (M,γ) is a balanced sutured manifold and suppose (M',γ') is the resulting sutured manifold after either a contact 1-handle or 2-handle attachment. For any spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(-M,-\gamma)$, suppose $\mathfrak{s}' \in \mathrm{Spin}^c(-M',-\gamma')$ is its extension corresponding to handle attachments. Then we have

$$C_{h^i}(SFH(-M,-\gamma,\mathfrak{s})) \subset SFH(-M',-\gamma',\mathfrak{s}'),$$

where $i \in \{1,2\}$.

Proof. We prove the claim on the chain level. After fixing a spin^c structure $\mathfrak{s}_0$ on (M,γ), we can identify $\mathrm{Spin}^c(M,\gamma)$ with $H^2(M,\partial M) \cong H_1(M)$. Moreover, we can represent the difference of two spin^c structures by a one-cycle in Proposition 4.3.

We can extend $\mathfrak{s}_0$ to a spin^c structure $\mathfrak{s}_0'$ on (M,γ) and identify $\mathrm{Spin}^c(M',\gamma')$ with $H_1(M')$. The inclusion $i: M \to M'$ induces a map

$$i_*: H_1(M) \to H_1(M').$$

For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the one cycle $\gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{y}}$ defined in Proposition 4.3 lies in the interior of M.

For a contact 1-handle, since the associated map C_{h^1} is tautological on intersection points, the homology class $i_*([\gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{y}}])$ characterizes the difference of spin^c structures on (M',γ') for $\boldsymbol{x}$ and $\boldsymbol{y}$.

For a contact 2-handle, since $\gamma_{\boldsymbol{x}\times\{c\}}$ is the union of multi-trajectory $\gamma_{\boldsymbol{x}}$ and the trajectory associated to c, we have

$$[\gamma_{\boldsymbol{x}\times\{c\}} - \gamma_{\boldsymbol{y}\times\{c\}}] = i_*([\gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{y}}]).$$

This implies the desired proposition. ■

Remark 4.17. The reader can compare Lemma 4.16 with Lemma 3.9. Note that when $H_1(M)$ has torsions, preserving the spin^c structures is stronger than preserving the gradings associated to an admissible surface.

Corollary 4.18. Suppose α is a bypass arc on a balanced sutured manifold (M,γ). Suppose (M,γ') is the resulting manifold after the bypass attachment along α. Then the bypass map ψ_α for *SFH* respects spin^c structures, that is, for any $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ and its extension $\mathfrak{s}' \in \mathrm{Spin}^c(M,\gamma')$, we have

$$\psi_\alpha(SFH(-M,-\gamma,\mathfrak{s})) \subset SFH(-M,-\gamma',\mathfrak{s}').$$

Proof. This follows directly from Lemma 4.16 by the fact that a bypass attachment is a composition of a contact 1-handle and 2-handle attachment. ■

Remark 4.19. By Corollary 4.18, if we consider the $\mathbb{Z}$-grading associated to an admissible surface S in Subsection 4.7, then the bypass exact triangle in Theorem 4.15 satisfies the similar grading shifting property to that in Lemma 3.3.

For twisted sutured homology, the map associated to a contact 2-handle is defined by the composition of the inverse of a contact 1-handle map and the cobordism map of a 0-surgery. The following proposition shows that we can define the map C_{h^2} for *SFH* in the same way.

Lemma 4.20 ([22]). Suppose (M, γ) is a balanced sutured manifold and (M', γ') is the resulting sutured manifold after a contact 2-handle attachment along $\mu \subset \partial M$. Let μ' be the framed knot obtained by pushing μ into the interior of M slightly, with the framing induced from ∂M. Suppose (N, γ_N) is the sutured manifold obtained from (M, γ) by a 0-surgery along μ'. Let

$$F_{\mu'} : SFH(-M, -\gamma) \to SFH(-N, -\gamma_N)$$

be the associated map. Let $D \subset N$ be the product disk that is the union of the annulus bounded by $\mu \cup \mu'$ and the meridian disk of the filling solid torus. Let

$$C_D : SFH(-N, -\gamma_N) \to SFH(-M', -\gamma')$$

be the map associated to the decomposition along D (i.e., the inverse of a contact 1-handle map). Then we have

$$C_{h^2,\mu} = C_D \circ F_{\mu'} : SFH(-M, -\gamma) \to SFH(-M', -\gamma').$$

Proof. Since all maps are well-defined on *SFH*, we can verify the claim by any diagram. Suppose (Σ, α, β) is a balanced diagram compatible with (M, γ). We note that the map associated to the 0-surgery along μ' may be achieved by first performing a compound stabilization and then computing a triangle map. The resulting diagram leaves an extra band that is deleted by C_D. By [50, Theorem 9.4], the claim then follows from a model computation in the stabilization region, as shown in Figure 12. ■

Combining the surgery exact triangle in Theorem 4.14 with Lemma 4.20, we obtain similar results in Lemma 3.6 for *SFH*.

Proposition 4.21. Consider the setups in Subsection 3.1. Suppose $T' = T \backslash \alpha = T_2 \cup \cdots \cup T_m$. Then for any $n \in \mathbb{N}$, there is an exact triangle

$$\begin{tikzcd} SFH(-M_T, -\Gamma_n) \arrow[rr] & & SFH(-M_T, -\Gamma_{n+1}) \arrow[dl, "F_{n+1}"] \\ & SFH(-M_{T'}, -\gamma_{T'}) \arrow[ul, "G_n"] & \end{tikzcd} \tag{4.7}$$

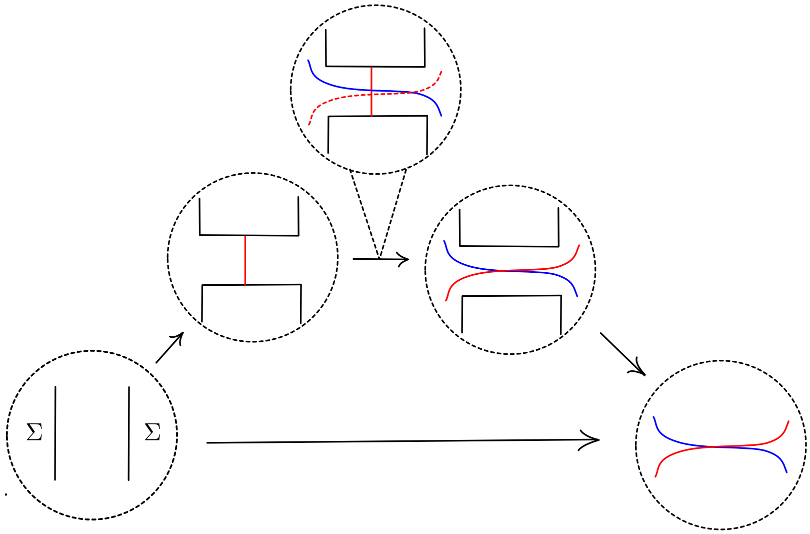

Fig. 12. Realizing the contact 2-handle map (bottom-most long arrow) as a composition of a compound stabilization (top), followed by a 4-dimensional 2-handle map (middle left), followed by a product disk map (middle right). A holomorphic triangle of the 2-handle map is indicated in the top subfigure.

The map F_{n+1} is induced by the contact 2-handle attachment along the meridian of α. Furthermore, we have commutative diagrams related to $\psi^n_{+,n+1}$ and $\psi^n_{-,n+1}$, respectively

$$\begin{tikzcd}
SFH(-M_T,-\Gamma_n) \arrow[rr, "\psi^n_{\pm,n+1}"] & & SFH(-M_T,-\Gamma_{n+1}) \\
& SFH(-M_{T'},-\gamma_{T'}) \arrow[ul, "G_n"] \arrow[ur, "G_{n+1}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
SFH(-M_T,-\Gamma_n) \arrow[rr, "\psi^n_{\pm,n+1}"] \arrow[dr, "F_n"] & & SFH(-M_T,-\Gamma_{n+1}) \arrow[dl, "F_{n+1}"'] \\
& SFH(-M_{T'},-\gamma_{T'}) &
\end{tikzcd}$$

Proof. It follows from the proof of Lemma 3.6. ■

5 Proof of Main Theorems

In this section, we prove Theorem 1.1 and Theorem 1.3 in the introduction.

Theorem 5.1 (Theorem 1.1). Suppose (M,γ) is a balanced sutured manifold. Suppose $H = H_1(M)$ and consider the (Turaev-type) torsion element $\tau(-M,-\gamma)$ in Theorem 4.13. Then we have

$$\chi_{\rm en}(\underline{SHG}(-M,-\gamma)) = \chi(SFH(-M,-\gamma)) = \tau(-M,-\gamma) \in \mathbb{Z}[H]/\pm H.$$

Proof. By Theorem 4.13, it suffices to prove the first equation.

First, we consider the case that (M,γ) is strongly balanced. By discussion in Subsection 4.1, we can construct a $\mathbb{Z}$-grading on SFH associated to an admissible surface $S \subset (M,\gamma)$. By discussion in Section 4, this $\mathbb{Z}$-grading also satisfies properties in Subsection 3.1 for the $\mathbb{Z}$-grading on $\underline{SHG}$ associated to S. Hence, for a vertical tangle T satisfies the conditions in Definition 3.23, we can define a vector space $\mathcal{SFH}_T(-M,-\gamma)$ similar to $\mathcal{SHG}_T(-M,-\gamma)$ in Definition 3.23. Note that for any $h_1 \in H_1(M)$, the summand of $\mathcal{SFH}_T(-M,-\gamma)$ is

$$SFH(-M_T,-\gamma_T,\boldsymbol{S},-\boldsymbol{i}')$$

with $j(\boldsymbol{\rho}^{-\boldsymbol{i}'}) = h$, where $\boldsymbol{S},\boldsymbol{\rho},\boldsymbol{i}'$ come from Definition 3.26, the gradings come from Definition 4.7, and $j: H_1(M_T) \to H_1(M)$ is the map induced by inclusion.

Similar to Proposition 3.25, there is an isomorphism

$$\mathcal{SFH}_T(-M,-\gamma) \cong SFH(-M,-\gamma). \tag{5.1}$$

Moreover, by the proofs of Lemma 3.6 and Proposition 3.17, the isomorphism in (5.1) is induced by contact 2-handle attachments along meridians of tangle components of T. Hence, by Lemma 4.16, the isomorphism in (5.1) respects spin^c structures. This implies that there exists $\mathfrak{s}_0 \in \mathrm{Spin}^c(-M,-\gamma)$, such that for any $h \in H_1(M)$, the summand of $\mathcal{SFH}_T(-M,-\gamma)$ corresponding to h is isomorphic to $SFH(-M,-\gamma,s_0+h)$. In particular, we have

$$\chi_{\rm en}(SFH(-M,-\gamma)) := j_*(\chi(\mathcal{SFH}_T(-M,-\gamma)) = \chi(SFH(-M,-\gamma)) \in \mathbb{Z}[H]/\pm H,$$

where $j_*: \mathbb{Z}[H_1(M_T)] \to \mathbb{Z}[H_1(M)]$.

By definition, the vector spaces $\mathcal{SFH}_T(-M,-\gamma)$ and $\mathcal{SHG}_T(-M,-\gamma)$ are direct summands of $SFH(-M_T,-\Gamma)$ and $\underline{SHG}(-M_T,-\Gamma)$ for some $\Gamma \subset \partial M_T$, respectively. By Lemma 3.20, the group $H_1(M_T)$ has no torsion. Hence, by (1.6), we have

$$\chi_{\rm gr}(\mathcal{SHG}_T(-M,-\gamma)) = \chi_{\rm gr}(\mathcal{SFH}_T(-M,-\gamma))$$
$$= \chi(\mathcal{SFH}_T(-M,-\gamma)) \in \mathbb{Z}[H_1(M_T)]/\pm H_1(M_T).$$

Thus, we have

$$\chi_{\rm en}(\underline{SHG}(-M,-\gamma)) = j_*(\chi_{\rm gr}(\mathcal{SHG}_T(-M,-\gamma)))$$
$$= \chi_{\rm en}(SFH(-M,-\gamma))$$
$$= \chi(SFH(-M,-\gamma)) \in \mathbb{Z}[H]/\pm H.$$

Then we consider the case that (M,γ) is not strongly balanced. As mentioned in Remark 4.5. If ∂M is not connected, we can construct a sutured manifold (M',γ') with connected boundary by attaching contact 1-handles (cf. [30, Remark 3.6]). The product disks in (M',γ') corresponding to these 1-handles are admissible surfaces, and only one summand in the associated $\mathbb{Z}$-grading is nontrivial. Hence, there is a canonical way to consider $\chi_{\rm en}(\underline{SHG}(-M',-\gamma'))$ as an element in $\mathbb{Z}[H_1(M)]/\pm H_1(M)$. We can consider $(-M',-\gamma')$ instead, and the above arguments about strongly balanced sutured manifolds apply to this case. ■

Proof of Theorem 1.3. We prove the theorem for $(-M,-\gamma)$, and it suffices to prove the case that (M,γ) is strongly balanced. Consider the construction of $\mathcal{SFH}_T(-M,-\gamma)$ in the proof of Theorem 5.1. We denote the monopole version of $\mathcal{SHG}_T(-M,-\gamma)$ by $\mathcal{SHM}_T(-M,-\gamma)$ and use it to provide a decomposition of $\underline{SHM}(-M,-\gamma)$.

By definition, the vector spaces $\mathcal{SFH}_T(-M,-\gamma)$ and $\mathcal{SHM}_T(-M,-\gamma)$ are direct summands of $SFH(-M_T,-\Gamma)$ and $\underline{SHM}(-M_T,-\Gamma)$ for some $\Gamma \subset \partial M_T$. By Lemma 3.20, the group $H_1(M_T)$ has no torsion. Hence, it suffices to prove the theorem for $(-M_T,-\Gamma)$.

We have

$$SHM(-M_T,-\Gamma) \cong \underline{SHM}(-M_T,-\Gamma) \cong SFH(-M_T,-\Gamma) \otimes \Lambda$$

with respect to the $H_1(M_T)$-grading induced by spin^c structures or admissible surfaces, where Λ is the Novikov ring over $\mathbb{Z}_2$. The first isomorphism comes from the construction of SHM and $\underline{SHM}$. The second isomorphism follows from [11, Theorem 3.1], which essentially depends on the isomorphism between $\check{HM}_\bullet$ and HF^+ for closed 3-manifolds (cf. [15, 37, 59]). The grading in the above isomorphism was discussed in [42, Corollary 3.42]. ∎

6 Knots With Small Instanton Knot Homology

In this section, we prove detection results about $\widehat{HFK}$ and KHI for null-homologous knots inside L-spaces.

6.1 Restrictions on Euler characteristics

In this subsection, we provide restrictions on Euler characteristics of $\widehat{HFK}$ and KHI. Since there is a canonical isomorphism between $\widehat{HFK}(Y,K)$ and $SFH(Y(K),\gamma_K)$, we do not distinguish them later.

Suppose (M,γ) is a balanced sutured manifold and $H = H_1(M;\mathbb{Z})$. In Definition 4.12, the Euler characteristic $\chi(SFH(M,\gamma))$ has an ambiguity of $\pm H$. When $M = Y(K)$ for a knot K in a rational homology sphere Y, we have $\partial M \cong T^2$. We can resolve the ambiguity of $\pm H$ as follows.

First, we resolve the sign ambiguity. Under the map $\mathbb{Z}[H] \to \mathbb{Z}[H_1(Y)]$ induced by inclusion, the Euler characteristic $\chi(SFH(M,\gamma))$ becomes

$$\pm \sum_{h\in H_1(Y)} h$$

by Theorem 4.13. Hence, we can fix the sign by choosing the one whose image is the positive one.

Then we resolve the ambiguity of H. We write $\mathrm{Spin}^c(Y,K)$ for $\mathrm{Spin}^c(Y(K),\gamma_K)$. First, there is a natural identification

$$SFH(Y(K),\gamma_K) \xrightarrow{\cong} SFH(Y(K),-\gamma_K),$$

where $-\gamma_K$ corresponds to $-K$, the knot obtained from K by reversing the orientation. Second, since $\partial Y(K) \cong T^2$, the suture $-\gamma_K$ is isotopic to the suture γ, we have a map

$$\iota : SFH(Y(K),\gamma_K) \xrightarrow{\cong} SFH(Y(K),-\gamma_K) \xrightarrow{=} SFH(Y(K),\gamma).$$

The square of this map is related to the basepoint moving map in [58, 68], though we do not use this fact.

If there is a spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(Y, K)$ so that $\widehat{HFK}(Y, K, \mathfrak{s})$ that is invariant under ι, then choose $\mathfrak{s}_0$ in Definition 4.12 so that

$$PD(\mathfrak{s} - \mathfrak{t}_0) = e,$$

where $e \in H$ is the identify element. If there is no summand that is invariant under ι, and suppose

$$\iota(\widehat{HFK}(Y, K, \mathfrak{s})) = \widehat{HFK}(Y, K, \mathfrak{s}')$$

for some $\mathfrak{s}, \mathfrak{s}' \in \mathrm{Spin}^c(Y, K)$, then we define $\chi(\widehat{HFK}(Y, K))$ as an element in $(\frac{1}{2}\mathbb{Z})[H]$ so that the group elements corresponding to $\widehat{HFK}(Y, K, \mathfrak{s})$ and $\widehat{HFK}(Y, K, \mathfrak{s}')$ are inverse elements. In particular, if $H \cong \mathbb{Z}$, then $(\frac{1}{2}\mathbb{Z})[H] \cong \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$. Note that this definition is independent of the choices of $\mathfrak{s}, \mathfrak{s}'$.

Definition 6.1. Suppose K is a knot in a rational homology sphere Y and Suppose $H = H_1(Y(K))$. When fixing the $\mathbb{Z}_2$-grading and the spin^c grading as above, the space $\widehat{HFK}(Y, K)$ is called the **canonical representative**. The corresponding spin^c grading is called the **absolute Alexander grading**. For the canonical representative of $\widehat{HFK}(Y, K)$, the Euler characteristic $\chi(\widehat{HFK}(Y, K))$ is a well-defined element in $\mathbb{Z}[H]$ or $(\frac{1}{2}\mathbb{Z})[H]$. For $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, define $\widehat{HFK}(Y, K, [\mathfrak{s}])$ as the direct summand of all $\widehat{HFK}(Y, K, \mathfrak{s}')$ with $\mathfrak{s}' \in \mathrm{Spin}^c(Y, K)$ and $\mathfrak{s}'$ extends to $\mathfrak{s}$ on Y. Then $\chi(\widehat{HFK}(Y, K, \mathfrak{s}))$ is also a well-defined element in $\mathbb{Z}[H]$ or $(\frac{1}{2}\mathbb{Z})[H]$.

Then we can state the main theorem of this subsection.

Theorem 6.2. Suppose K_1 and K_2 be two knots in a rational homology sphere Y with $[K_1] = [K_2] \in H_1(Y)$. For $i = 1, 2$, suppose m_i is the meridian of K_i. Then there exists an isomorphism $\phi : H_1(Y(K_1); \mathbb{Z}) \cong H_1(Y(K_2); \mathbb{Z})$ so that $\phi([m_1]) = [m_2]$. Using the isomorphism ϕ, we write $H_1(Y(K_i); \mathbb{Z})$ as H and write $[m_i]$ as $[m] \in H$.

Consider the canonical representative of $\widehat{HFK}(Y, K_i)$. Then for any $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, there exists a Laurent polynomial $f_{\mathfrak{s}}(x) \in \mathbb{Z}[x, x^{-1}]$ and an element $h_{\mathfrak{s}} \in H$ such that

$$\chi(\widehat{HFK}(Y, K_1, [\mathfrak{s}])) - \chi(\widehat{HFK}(Y, K_2, [\mathfrak{s}])) = ([m] - 1)^2 f_{\mathfrak{s}}([m]) h_{\mathfrak{s}},$$

where both sides are elements in $\mathbb{Z}[H]$ or $(\frac{1}{2}\mathbb{Z})[H]$.

Note that Theorem 6.2 is a generalization of [66, Theorem 5.8]. Indeed, the proof of [66, Theorem 5.8] applies without essential change. In the following, we prove generalizations of lemmas in [66, Section 5] and then sketch the proof of Theorem 6.2.

Lemma 6.3. Suppose K_1 and K_2 be two knots in a rational homology sphere Y with $[K_1]=[K_2]\in H_1(Y)$. Then there exists an isomorphism $\phi : H_1(Y(K_1);\mathbb{Z})\cong H_1(Y(K_2);\mathbb{Z})$ so that $\phi([m_1])=[m_2]$.

Proof. For a manifold M, let $T_1(M)$ be the torsion subgroup of $H_1(M)$ and let $B_1(M)=H_1(M)/T_1(M)$. By [14, Theorem 3.1], since $[K_1]=[K_2]\in H_1(Y)$, there is a subgroup B of $B_1(Y(K_1\cup K_2)$ so that for $i=1,2$, the map

$$j_i : B_1(Y(K_1\cup K_2))\to B_1(Y(K_i))$$

induced by the injection is an isomorphism on B. Moreover, the map

$$k_i : T_1(Y(K_1\cup K_2))\to T_1(Y(K_i))$$

induced by injection is an isomorphism. Since $H_1(M)\cong B_1(M)\oplus T_1(M)$ for any manifold M, we have an isomorphism

$$\begin{aligned}\phi_0=(j_2\circ j_1^{-1},k_2\circ k_1^{-1}) : H_1(Y(K_1)) &\cong B_1(Y(K_1))\oplus T_1(Y(K_1))\\ &\cong B_1(Y(K_2))\oplus T_1(Y(K_2))\cong H_1(Y(K_2)).\end{aligned}$$

Moreover, if

$$l_i : H_1(Y(K_i))\to H_1(Y)$$

is the map induced by injection, then $l_1=\phi_0\circ l_2$.

For $i=1,2$, consider the long exact sequence about the pair $(Y,N(K_i))$:

$$H^1(Y)\to H^1(N(K_i))\to H^2(Y,N(K_i))\to H^2(Y)\to H^2(K_i)=0.$$

Since Y is a rational homology sphere,

$$H^1(Y)\cong \mathrm{Hom}_{\mathbb{Z}}(H_1(Y),\mathbb{Z})=0.$$

By excision theorem and the Poincaré duality, we have

$$H^2(Y, N(K_i)) \cong H^2(Y(K_i), \partial Y(K_i)) \cong H_1(Y(K_i)) \text{ and } H^2(Y) \cong H_1(Y).$$

Under the Poincaré duality, the image of

$$H^1(N(K_i)) \cong H_1(N(K_i), \partial N(K_i)) \cong \mathbb{Z}$$

in $H_1(Y(K_i))$ is $[m_i]$. Since $l_1 = \phi \circ l_2$, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & H_1(Y(K_1)) \arrow[r, "l_1"] \arrow[d, "\phi_0"] & H_1(Y) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & H_1(Y(K_2)) \arrow[r, "l_2"] & H_1(Y) \arrow[r] & 0
\end{tikzcd}$$

Hence, $\phi_0([m_1]) = [m_2]^{\pm 1}$. If $\phi_0([m_1]) = [m_2]$, let $\phi = \phi_0$. If $\phi_0([m_1]) = [m_2]^{-1}$, let $\phi = \phi_0 \circ \epsilon$, where ϵ maps an element to its inverse. Then $\phi : H_1(Y(K_1)) \to H_1(Y(K_2))$ is an isomorphism and $\phi([m_1]) = ([m_2])$. ■

Lemma 6.4. Suppose G is an abelian group and g_0 is an element in G. The quotient map $G \to G/(g_0)$ induces a map on the group ring

$$\mathrm{pr} : \mathbb{Z}[G] \to \mathbb{Z}[G/(g_0)].$$

Then the kernel of pr is generated by $1 - g_0$.

Proof. We can regard element in $\mathbb{Z}[G]$ as a function $f : G \to \mathbb{Z}$ that maps $g \in G$ to the coefficient of g. Note that $f(g) \neq 0$ for finitely many g. If $f \in ker(\mathrm{pr})$, then

$$h(g) = \sum_{k \in \mathbb{N}} f(g g_0^{-k})$$

is also a function $G \to \mathbb{Z}$ that is nonvanishing for finitely many g. It is straightforward to check that $h = (1 - g_0)f$ as elements in $\mathbb{Z}[G]$. ■

Proof of Theorem 6.2. The first part of this theorem is just Lemma 6.3. Write $H = H_1(Y(K_i))$ and $[m] = [m_i]$. By Theorem 4.13, we have

$$\chi(\widehat{HFK}(Y, K_i)) = (1 - [m])\tau(K_i) \text{ in } \mathbb{Z}[H] \text{ or } (\frac{1}{2}\mathbb{Z})[H].$$

A priori, the Turaev torsion $\tau(Y(K_i))$ is not in $\mathbb{Z}[H]$, but the difference $\tau(Y(K_1)) - \tau(Y(K_2))$ is. By Lemma 6.4, we can apply the proof of [66, Lemma 5.5] to the case where Y is a rational homology sphere. Note that we use the fact $b_1(Y(K_i)) = 1$ in that proof. From [66, Lemma 5.5], we have

$$\tau(Y(K_1)) - \tau(Y(K_2)) = (1 - [m])g \text{ in } \mathbb{Z}[H] \text{ or } (\frac{1}{2}\mathbb{Z})[H] \text{ for some} g \in \mathbb{Z}[H].$$

The ambiguity of $\pm H$ in the statement of [66, Lemma 5.5] is resolved because we consider absolute Alexander gradings on $\widehat{HFK}(Y, K_i)$. Then we have

$$\chi(\widehat{HFK}(Y, K_1)) - \chi(\widehat{HFK}(Y, K_2)) = (1 - [m])(\tau(Y(K_1)) - \tau(Y(K_2))) = (1 - [m])^2 g. \tag{6.1}$$

Suppose $H_1(Y; \mathbb{Z}) = \{s_1, \ldots, s_p\}$. Then the element g can be written as the sum

$$g = \sum_{j=1}^{p} g_j,$$

where g_j contains terms that are in the preimage of $s_j \in H_1(Y; \mathbb{Z})$ under the map

$$H \to H/m \cong H_1(Y; \mathbb{Z}).$$

For any j, there exists a Laurent polynomial $f_j(x)$ and an element $\tilde{s}_j \in H$ such that $g_j = f_j([m])\tilde{s}_j$. Since $\mathrm{Spin}^c(Y)$ is an affine space on $H_1(Y)$. Thus, Equation (6.1) can be decomposed with respect to $\mathrm{Spin}^c(Y)$, where $h_{\mathfrak{s}}$ corresponds to some $\tilde{s}_j$. ■

Finally, we deal with instanton knot homology.

Definition 6.5. Suppose K is a knot in a rational homology sphere Y and suppose $H = H_1(Y(K))$. Similar to the way for $\widehat{HFK}(Y, K)$, we can fix the $\mathbb{Z}_2$-grading and the H-grading from the enhanced Euler characteristic on $KHI(Y, K)$. When gradings are fixed, the space $KHI(Y, K)$ is called also the **canonical representative** and the corresponding H-grading is also called the **absolute Alexander grading**. For any element $s \in H_1(Y)$, let $[s] \subset$

$H_1(Y(K))$ be the set of preimages of s under the map $H \to H_1(Y)$ and define

$$KHI(Y,K,[s]) := \bigoplus_{h\in[s]} KHI(Y,K,h).$$

Then $\chi(KHI(Y,K,[s])$ is also a well-defined element in $\mathbb{Z}[H]$ or $(\frac{1}{2}\mathbb{Z})[H]$.

By Theorem 6.2 and Equation (1.1), we have the following corollary.

Corollary 6.6. Suppose K_1 and K_2 be two knots in a rational homology sphere Y with $[K_1] = [K_2] \in H_1(Y)$. For $i = 1, 2$, suppose m_i is the meridian of K_i. Using the isomorphism ϕ in Lemma 6.3, we write $H_1(Y(K_i);\mathbb{Z})$ as H and write $[m_i]$ as $[m] \in H$.

Consider the canonical representative of $KHI(Y,K_i)$. Then for any $s \in H_1(Y)$, there exists a Laurent polynomial $f_s(x) \in \mathbb{Z}[x, x^{-1}]$ and an element $h_s \in H$ such that

$$\chi(KHI(Y,K_1,[s])) - \chi(KHI(Y,K_2,[s])) = ([m]-1)^2 f_s([m]) h_s,$$

where both sides are elements in $\mathbb{Z}[H]$ or $(\frac{1}{2}\mathbb{Z})[H]$.

6.2 Detection results

In this subsection, we use Theorem 6.2 and Corollary 6.6 to prove detection results in the introduction.

Convention. Throughout this subsection, we suppose K is a knot in a rational homology sphere Y and suppose $H = H_1(Y(K))$. Moreover, we consider canonical representatives of $\widehat{HFK}(Y,K)$ and $KHI(Y,K)$ as in Definition 6.1 and Definition 6.5. For simplicity, we write also write $(\frac{1}{2}\mathbb{Z})[H]$ as $\mathbb{Z}[H]$.

First, we prove some lemmas.

Lemma 6.7 (). Suppose $K \subset Y$ is a knot so that $[K] = 0 \in H_1(Y)$. Then there exists a canonical isomorphism

$$H_1(Y(K)) \cong \mathbb{Z} \oplus H_1(Y),$$

where the meridian of K represents the generator of $\mathbb{Z}$.

Proof. The isomorphism is induced by pairing with a Seifert surface of K. ■

Hence we can write elements in $H_1(Y(K))$ as $[m]^n \cdot s$ for $s \in H_1(Y)$ and $n \in \mathbb{Z}$.

Lemma 6.8. Suppose Y is an instanton L-space and $U \subset Y$ is the unknot. Then for any $s \in H_1(Y)$, we have

$$KHI(Y, U, [s]) \cong \mathbb{C} \text{ and } \chi(KHI(Y, U, [s])) = s \in \mathbb{Z}[H].$$

Proof. The result $KHI(Y, U, [s]) \cong \mathbb{C}$ follows directly from Equation (1.1) and the following isomorphisms:

$$KHI(Y, U) \cong I^{\sharp}(Y) \cong \mathbb{C}^{|H_1(Y)|}.$$

The result $\chi(KHI(Y, U, [s])) = 1$ follows from the fact that $g(U) = 0$ and KHI detects the genus of the knot [36, Proposition 7.16]. ■

Lemma 6.9. Suppose Y is an instanton L-space, and $K \subset Y$ is a knot so that $[K] = 0 \in H_1(Y)$. Suppose m is the meridian of K. Then for any $s \in H_1(Y)$, the element

$$\chi(KHI(Y, K, [s])) - s \in \mathbb{Z}[H]$$

has a factor $([m] - 1)^2$. See Definition 6.5 for the definition of $\chi(KHI(Y, K, [s]))$.

Proof. Since the unknot is also null-homologous, this lemma follows directly from Corollary 6.6 and Lemma 6.8. ■

Then we prove the detect results in the introduction.

Proof of Theorem 1.5. It is clear that

$$\dim_{\mathbb{C}} KHI(Y, K) = \dim_{\mathbb{C}} I^{\sharp}(Y)$$

when K is the unknot in Y. Now suppose

$$\dim_{\mathbb{C}} KHI(Y, K) = \dim_{\mathbb{C}} I^{\sharp}(Y)$$

and we will show that K must be the unknot. For any $s \in H_1(Y)$, Lemma 6.9 implies that

$$\chi(KHI(Y, K, [s])) \neq 0$$

and hence $KHI(Y, K, [s]) \neq 0$. From the assumption, we have

$$\dim_{\mathbb{C}} KHI(Y, K) = \dim_{\mathbb{C}} I^{\sharp}(Y) = |H_1(Y)|.$$

Thus, we must have

$$KHI(Y, K, [s]) \cong \mathbb{C} \text{ and } \chi(KHI(Y, K, [s])) = [m]^n \cdot s \in \mathbb{Z}[H],$$

where m is the meridian of K and n is some integer. Applying Lemma 6.9 again, we know that n must be 0. Since KHI detects the genus of the knot [36, Proposition 7.16], we know that $g(K) = 0$, which implies K is the unknot. ■

Proof of Theorem 1.8. Applying Lemma 6.9, for any $s \in H_1(Y)$, we have

$$\chi(KHI(Y, K, [s])) \neq 0.$$

Since $\chi(KHI(Y, K, [s]))$ and $\dim_{\mathbb{C}} KHI(Y, K, [s])$ have the same parity, we conclude that there exists $s_0 \in H_1(Y)$ so that

$$KHI(Y, K, [s_0]) \cong \mathbb{C}^3$$

and for any $s \neq s_0$,

$$KHI(Y, K, [s]) \cong \mathbb{C}$$

Applying Lemma 6.9 again, for any $s \neq s_0$, we know that

$$\chi(KHI(Y, K, [s])) = s \in \mathbb{Z}[H].$$

For s_0, we know that

$$\chi(KHI(Y, K, [s_0])) - s_0 \in \mathbb{Z}[H]$$

has a factor $([m] - 1)^2$ and

$$\|\chi(KHI(Y, K, [s_0]))\| \leq 3,$$

where m is the meridian of K and the norm is defined before Example 1.4.

Hence, there are only two possibilities.

Case 1. $\|\chi(KHI(Y,K,[s_0]))\| = 1$ and we then conclude that

$$\chi(KHI(Y,K,[s_0])) = [m]^n \cdot s_0 \in \mathbb{Z}[H].$$

Note that $\|\chi(KHI(Y,K,[s_0]))\| = 1$ implies that there is a 2-dimensional summand of $KHI(Y,K)$ whose Euler characteristic is zero. Hence there are further two cases:

Case 1.1 $KHI(Y,K,[s_0])$ is supported in two different Alexander gradings. By assumption, we know that $KHI(Y,K,[s_0])$ has a 1-dimensional summand at the Alexander grading n and has a 2-dimensional summand at the Alexander grading n' for some $n' \neq n$. This contradicts the fact that $KHI(Y,K)$ is symmetric with respect to the Alexander grading.

Case 1.2. $KHI(Y,K,[s_0])$ is support solely in the Alexander grading n. By symmetry of $KHI(Y,K)$, we must have $n = 0$. Since KHI detects the genus of the knot, we know K is an knot, which contradicts Theorem 1.5 since $\dim_{\mathbb{C}} KHI(Y,K) = \dim_{\mathbb{C}} I^{\sharp}(Y)+2$.

Case 2. $\|\chi(KHI(Y,K,[s_0]))\| = 3$. By symmetry on $KHI(Y,K)$, there exists $n \in \mathbb{N}_+$so that

$$\chi(KHI(Y,K,[s_0])) = ([m]^n - 1 + [m]^{-n}) \cdot s_0 \in \mathbb{Z}[H].$$

by [36, Proposition 7.16 and Corollary 7.19], we know that K is fibred of genus n. By [12, Theorem 1.7], we know that $n = 1$. Hence K is a genus-one-fibred knot. ■

The proofs of the following theorems are similar to those of Theorem 1.5 and Theorem 1.8, but using Theorem 6.2 and [13, Theorem 1.1] instead of Corollary 6.6 and [12, Theorem 1.7].

Theorem 6.10. Suppose K is a null-homologous knot in a rational homology sphere Y. If

$$\dim_{\mathbb{F}_2} \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|, \tag{6.2}$$

then K is the unknot if and only if

$$\dim_{\mathbb{F}_2} \widehat{HFK}(Y,K) = \dim_{\mathbb{F}_2} \widehat{HF}(Y). \tag{6.3}$$

Theorem 6.11. Suppose K is a null-homologous knot in a rational homology sphere Y. If

$$\dim_{\mathbb{F}_2} \widehat{HFK}(Y,K) = \dim_{\mathbb{F}_2} \widehat{HF}(Y) + 2 = |H_1(Y;\mathbb{Z})| + 2, \tag{6.4}$$

then K must be a genus-one-fibred knot.

Remark 6.12. Baldwin [4] classified L-spaces that contain null-homologous genus-one-fibred knots. He also computed knot Floer homologies of such knots, which only depend on their Alexander polynomials. The techniques in his classification involve the minus chain complex [50] and the mapping cone formula [54, 55], which is not available in instanton theory yet. For knots in lens spaces, there are more results about genus-one-fibred knots [1, 3, 44].

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References

[1] Baker, K. L. "Counting genus one fibered knots in lens spaces." *Michigan Math. J.* 63, no. 3 (2014): 553–69.

[2] Baker, K. L., J. Elisenda Grigsby, and M. Hedden. "Grid diagrams for lens spaces and combinatorial knot Floer homology." *Int. Math. Res. Not.* 2008 (2008). http://10.1093/imrn/rnn024.

[3] Baker, K. L., J. E. Johnson, and E. A. Klodginski. "Tunnel number one, genus-one fibered knots." *Commun. Anal. Geom.* 17, no. 1 (2009): 1–16. http://10.4310/CAG.2009.v17.n1.a1.

[4] Baldwin, J. A. "Heegaard Floer homology and genus one, one boundary component open books." *J. Topol.* 1, no. 4 (2008): 963–92. http://10.1112/jtopol/jtn029.

[5] Baldwin, J. A., H. Ying, and S. Sivek. "Khovanov homology and the cinquefoil." Preprint arXiv:2105.12102, v1, 2021.

[6] Baldwin, J. A., Z. Li, and F. Ye. "Sutured instanton homology and Heegaard diagrams." Preprint arXiv: 2011.09424, v1, 2020.

[7] Baldwin, J. A. and S. Sivek. " Naturality in sutured monopole and instanton homology." *J. Differ. Geom.* 100, no. 3 (2015): 395–480.

[8] Baldwin, J. A. and S. Sivek. "A contact invariant in sutured monopole homology." *Forum Math. Sigma* 4 (2016): e12, 82. http://10.1017/fms.2016.11.

[9] Baldwin, J. A. and S. Sivek. "Instanton Floer homology and contact structures." *Selecta Math. (N.S.)* 22, no. 2 (2016): 939–78. http://10.1007/s00029-015-0206-x.

[10] Baldwin, J. A. and S. Sivek. "Instanton and L-space surgeries." Preprint arXiv:1910.13374, v1, 2019.

[11] Baldwin, J. A. and S. Sivek. "On the equivalence of contact invariants in sutured Floer homology theories." Preprint arXiv:1601.04973, v3, 2020.

[12] Baldwin, J. A. and S. Sivek. "Khovanov homology detects the trefoils." *Duke Math. J.* 174, no. 4 (2022): 885–956.

[13] Baldwin, J. A. and D. S. Vela-Vick. "A note on the knot Floer homology of fibered knots." *Algebr. Geom. Topol.* 18, no. 6 (2018): 3669–90. http://10.2140/agt.2018.18.3669.

[14] Brody, E. J. "The topological classification of the lens spaces." *Ann. Math.* 71, no. 1 (1960): 163. http://10.2307/1969884.

[15] Colin, V., P. Ghiggini, and K. Honda. "The equivalence of Heegaard Floer homology and embedded contact homology III: from hat to plus." Preprint arXiv:1208.1526, v2, 2017.

[16] Culler, M., N. M. Dunfield, G. Matthias, and J. R. Weeks. "SnapPy, a computer program for studying the geometry and topology of 3-manifolds." http://snappy.computop.org, 2021.

[17] Etnyre, J. B., D. S. Vela-Vick, and R. Zarev. "Sutured Floer homology and invariants of Legendrian and transverse knots." *Geom. Topol.* 21, no. 3 (2017): 1469–582. http://10.2140/gt.2017.21.1469.

[18] Friedl, S., A. Juhász, and J. Rasmussen. "The decategorification of sutured Floer homology." *J. Topol.* 4, no. 2 (2009): 431–78.

[19] Gabai, D. "Foliations and the topology of 3-manifolds." *J. Differ. Geom.* 18, no. 3 (1983): 445–503.

[20] Ghosh, S. and Z. Li. "Decomposing sutured monopole and instanton Floer homologies." Preprint arXiv:1910.10842, v2, 2019.

[21] Ghosh, S., Z. Li, and C.-M. Michael Wong. "Tau invariants in monopole and instanton theories." Preprint arXiv:1910.01758, v3, 2019.

[22] Ghosh, S. and I. Zemke. "Connected sums and the torsion order in instanton knot homology." In press.

[23] Goda, H., H. Matsuda, and T. Morifuji. "Knot Floer homology of (1,1)-knots." *Geom. Dedicata* 112, no. 1 (2005): 197–214. http://10.1007/s10711-004-5403-2.

[24] Greene, J. E., S. Lewallen, and F. Vafaee. "(1,1) L-space knots." *Compos. Math.* 154, no. 5 (2018): 918–33. http://10.1112/S0010437X17007989.

[25] Honda, K. "Contact structures, Heegaard Floer homology and triangulated categories." In press.

[26] Honda, K. "On the classification of tight contact structures I." *Geom. Topol.* 4 (2000): 309–68. http://10.2140/gt.2000.4.309.

[27] Honda, K. "Gluing tight contact structures." *Duke Math. J.* 115, no. 3 (2002): 435–78.

[28] Honda, K., W. H. Kazez, and G. Matić. "Contact structures, sutured Floer homology and TQFT." Preprint arXiv:0807.2431, v1, 2008.

[29] Juhász, A. "Holomorphic discs and sutured manifolds." *Algebr. Geom. Topol.* 6, no. 3 (2006): 1429–57. http://10.2140/agt.2006.6.1429.

[30] Juhász, A. "Floer homology and surface decompositions." *Geom. Topol.* 12, no. 1 (2008): 299–350. http://10.2140/gt.2008.12.299.

[31] Juhász, A. " The sutured Floer homology polytope." *Geom. Topol.* 14, no. 3 (2010): 1303–54. 10.2140/gt.2010.14.1303.

[32] Juhász, A. "Cobordisms of sutured manifolds and the functoriality of link Floer homology." *Adv. Math.* 299 (2016): 940–1038. http://10.1016/j.aim.2016.06.005.

[33] Juhász, A., D. P. Thurston, and I. Zemke. "Naturality and mapping class groups in Heegaard Floer homology." *Mem. Amer. Math. Soc.* 273, no. 1338 (2021): v+174 pp.

[34] Juhász, A. and I. Zemke. "Contact handles, duality, and sutured Floer homology." *Geom. Topol.* 24, no. 1 (2020): 179–307. http://10.2140/gt.2020.24.179.

[35] Kronheimer, P. B. and T. S. Mrowka. "Instanton Floer homology and the Alexander polynomial." *Algebr. Geom. Topol.* 10, no. 3 (2010): 1715–38. http://10.2140/agt.2010.10.1715.

[36] Kronheimer, P. B. and T. S. Mrowka. "Knots, sutures, and excision." *J. Differ. Geom.* 84, no. 2 (2010): 301–64.

[37] Kutluhan, C., Y.-J. Lee, and C. H. Taubes. "HF = HM I: Heegaard Floer homology and Seiberg–Witten Floer homology." *Geom. Topol.* 24, no. 6 (2020): 2829–54. http://10.2140/gt.2020.24.2829.

[38] Leigon, R. and F. Salmoiraghi. "Equivalence of contact gluing maps in sutured Floer homology." Preprint arXiv: 2005.04827, v1, 2020.

[39] Lekili, Y. "Heegaard–Floer homology of broken fibrations over the circle." *Adv. Math.* 244 (2013): 268–302. http://10.1016/j.aim.2013.05.013.

[40] Li, Z. "Gluing maps and cobordism maps in sutured monopole and instanton Floer theories." *Algebr. Geom. Topol.* 21, no. 6 (2021): 3019–71. http://10.2140/agt.2021.21.3019.

[41] Li, Z. "Knot homologies in monopole and instanton theories via sutures." *J. Symplectic Geom.* 19, no. 6 (2021): 1339–420. http://10.4310/JSG.2021.v19.n6.a2.

[42] Li, Z. and F. Ye. " Instanton Floer homology, sutures, and Euler characteristics." Preprint arXiv: 2011.09424, v2, 2021.

[43] Li, Z. and F. Ye. "Instanton Floer homology, sutures, and Heegaard diagrams." *J. Topol.* 15, no. 1 (2022): 39–107. http://10.1112/topo.12218.

[44] Morimoto, K. "Genus one fibered knots in lens spaces." *J. Math. Soc. Japan* 41, no. 1 (1989): 81–96.

[45] Ni, Y. "A note on knot Floer homology and fixed points of monodromy." Preprint arXiv:2106.03884, v1, 2021.

[46] Ni, Y. and X. Zhang. "Characterizing slopes for torus knots, II." Preprint arXiv:2106.02806, v1, 2021.

[47] Ozbagci, B. " Contact handle decompositions." *Topol. Appl.* 158, no. 5 (2011): 718–27. http://10.1016/j.topol.2011.01.020.

[48] Ozsváth, P. S. and Z. Szabó. "Holomorphic disks and knot invariants." *Adv. Math.* 186, no. 1 (2004): 58–116. http://10.1016/j.aim.2003.05.001.

[49] Ozsváth, P. S. and Z. Szabó. "Holomorphic disks and three-manifold invariants: properties and applications." *Ann. Math.* 159 (2004): 1159–245. http://10.4007/annals.2004.159.1159.

[50] Ozsváth, P. S. and Z. Szabó. "Holomorphic disks and topological invariants for closed three-manifolds." *Ann. Math. (2)* 159, no. 3 (2004): 1027–158.

[51] Ozsváth, P. S. and Z. Szabó. "On the Heegaard Floer homology of branched double-covers." *Adv. Math.* 194, no. 1 (2005): 1–33. http://10.1016/j.aim.2004.05.008.

[52] Ozsváth, P. S. and Z. Szabó. "Holomorphic triangles and invariants for smooth four-manifolds." *Adv. Math.* 202 (2006): 326–400. http://10.1016/j.aim.2005.03.014.

[53] Ozsváth, P. S. and Z. Szabó. "Lectures on Heegaard Floer homology." In *Floer Homology, Gauge Theory, and Low Dimensional Topology*. Clay Mathematics Proceedings, vol. 5, 29–70. American Mathematical Society, Providence, RI, 2006.

[54] Ozsváth, P. S. and Z. Szabó. "Knot Floer homology and integer surgeries." *Algebr. Geom. Topol.* 8, no. 1 (2008): 101–53. http://10.2140/agt.2008.8.101.

[55] Ozsváth, P. S. and Z. Szabó. "Knot Floer homology and rational surgeries." *Algebr. Geom. Topol.* 11, no. 1 (2011): 1–68. http://10.2140/agt.2011.11.1.

[56] Rasmussen, J. "Floer homology and knot complements." Preprint arXiv:math/0306378, v1, 2003.

[57] Rasmussen, J. and S. D. Rasmussen. "Floer simple manifolds and L-space intervals." *Adv. Math.* 322 (2017): 738–805. http://10.1016/j.aim.2017.10.014.

[58] Sarkar, S. "Moving basepoints and the induced automorphisms of link Floer homology." *Algebr. Geom. Topol.* 15, no. 5 (2015): 2479–515. http://10.2140/agt.2015.15.2479.

[59] Taubes, C. H. "Embedded contact homology and Seiberg–Witten Floer cohomology I." *Geom. Topol.* 14, no. 5 (2010): 2497–581. http://10.2140/gt.2010.14.2497.

[60] Turaev, V. *Torsions of 3-Dimensional Manifolds*. Birkhäuser, Basel, 2002, http://10.1007/978-3-0348-7999-6.

[61] Wang, J. "The cosmetic crossing conjecture for split links." Preprint arXiv:2006.01070, v1, 2020.

[62] Williams, M. J. "On nonsimple knots in lens spaces with tunnel number one." Preprint arXiv:0908.1765, v1, 2009.

[63] Xie, Y. and B. Zhang. "Instanton Floer homology for sutured manifolds with tangles." Preprint arXiv:1907.00547, v2, 2019.

[64] Xie, Y. and B. Zhang. "On meridian-traceless su(2)-representations of link groups." Preprint-arXiv:2104.04839, v2, 2021.

[65] Ye, F. "Data and codes to accompany this paper." *Harvard Dataverse* (2020). https://doi.org/10.7910/DVN/GLFLHI.

[66] Ye, F. "Constrained knots in lens spaces." Preprint arXiv:2007.04237, v2, 2021.

[67] Zarev, R. "Joining and gluing sutured Floer homology." Preprint arXiv:1010.3496, v1, 2010.

[68] Zemke, I. "Quasistabilization and basepoint moving maps in link Floer homology." *Algebr. Geom. Topol.* 17, no. 6 (2017): 3461–518. http://10.2140/agt.2017.17.3461.